퍼지관계방정식의 해의 관계성

On the solutions $U^{\dagger}$ and $U_{\dagger}$ of fuzzy relation equation

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Abstract

The purpose of this paper is to investigate solutions $U_{\dagger}$ and $U^{\dagger}$ for the fuzzy relation equation $R \circ U = T$ in cases of $R < T$, $R \leq T$, and $R = T$, when R is irreflexive, $U_{\dagger}(x_i, x_k) = \bigwedge[R(x_i, x_j) \ll T(x_j, x_k)]$, $U^{\dagger}(x_i, x_k) = \bigwedge[R(x_j, x_i) \rightarrow T(x_j, x_k)]$.

1. Introduction

Let us consider the lattice $L = ([0, 1], \vee, \wedge, \rightarrow, \ll)$, where

$$a \vee b = \max(a, b),$$
$$a \wedge b = \min(a, b),$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$
$$a \ll b = \begin{cases} 1 & \text{if } a \geq b, \\ a & \text{if } a < b. \end{cases}$$

2. Preliminaries

The existence of solution of the relation equation

$$R \circ U = T$$

(with unknown relation U and given relations R, T) was characterized by Sanchez [1].

We need the following definitions and properties.

Let X be non-empty a finite set , card(X) = n .

Definition 2.1

A *fuzzy binary relation* on X and Y is a fuzzy subset R on $X \times Y$.

We are only interested in the case in which $X = Y$.

Definition 2.2

Suppose R and U are two fuzzy relation on X.

$$(R \circ U)(x_i, x_k) = \bigvee [R(x_i, x_j) \wedge T(x_j, x_k)] \ , \forall x_i, x_j, x_k \in X,$$

where $\circ$ operation is called a *sup-inf composition*.

Definition 2.3

We say that I is called an *identity relation* on X if $R \circ I = I \circ R = R$,

where $I(x, y) = 1$ if $x = y$,

0 if $x \neq y$.

Definition 2.4

1) A fuzzy relation R is said to *reflexive* if $I \leq R$.

2) A fuzzy relation R is *irreflexive* iff $I \wedge R = \emptyset$.

3) If $R \circ R \leq R$,then R is called *transitive*.

Theorem 2.5 [2]

Equation $R \circ U = T$ has solutions iff $R \circ U^{\dagger} = T$, where

$$U^{\dagger}(x, z) = \bigwedge [R(y, x) \rightarrow T(y, z)] \qquad \forall x, y, z \in X .$$

If $R \circ U = T$ has solutions , then the above formula gives the greatest one.

In general, we always have $R \circ U^{\dagger} \leq T$.

3. Result

Theorem 3.1

Let R be irreflexive.

1] If $R < T$,then $U_{\dagger} = \emptyset$.

2] If $R = T$ and $R(x_i, x_j) \neq 0$, where $i \neq j$,then $U_{\dagger} \leq I = U^{\dagger}$.

Proof.

1] Let $R(x_i, x_j) = [\, r_{ij} \,]$, $T(x_j, x_k) = [\, t_{jk} \,]$, $\forall\, r_{ij}, t_{jk} \in [0, 1]$.

Since $R < T$ and R is irreflexive,

$$\begin{aligned} U_{\dagger}(x_i, x_k) &= \bigwedge_j [R(x_i, x_j) \ll T(x_j, x_k)] \\ &= \bigwedge [\, r_{i1} \ll t_{1k} \,,\, r_{i2} \ll t_{2k} \,,\, \cdots,\, r_{ii} \ll r_{ik} \,,\, \cdots,\, r_{in} \ll t_{nk} \,] \\ &= 0 \qquad \text{for all } i, j, k \leq n. \end{aligned}$$

2] Let $U_{\dagger}(x_i, x_k) = \bigwedge [R(x_i, x_j) \ll R(x_j, x_k)]$ --------------(1.1),

$U^{\dagger}(x_i, x_k) = \bigwedge [R(x_j, x_i) \rightarrow R(x_j, x_k)]$ -------------(1.2).

For any $i, j, k \leq n$, the right -hand member of (1.1) is

$$\bigwedge [\, r_{i1} \ll r_{1k}, r_{i2} \ll r_{2k}, \cdots, r_{in} \ll r_{nk}].$$

I) Let $i = k$, we find $U_{\dagger}(x_i, x_k) = r_{ih}$ if $r_{ih} < r_{hi}$,

1 if $r_{ih} \geq r_{hi}$, $\forall h \leq n$. --(1.3)

In case of $i \neq k$, $U_{\dagger} \equiv R \ll R$

$$\begin{aligned} &= \bigwedge_j [\, r_{ij} \ll r_{jk}] \\ &= \bigwedge [\, r_{i1} \ll r_{1k}, r_{i2}, \ll r_{2k}, \cdots, r_{in} \ll r_{nk}] \end{aligned}$$ ----(1.4)

The right-hand member of (1.4) contains $r_{ii} \ll r_{ik}$. Since $R = T$, $r_{ik} \neq 0$, $\forall i \neq k$, $U_{\dagger} = 0$. Thus $U_{\dagger} = \{0 \,, r_{ih} \,, 1 \}$.

ii) For any $i, j, k \leq n$, the right-hand member of (1.2) is

$$\bigwedge_j [\, r_{ji} \rightarrow r_{jk}] = \bigwedge [\, r_{1i} \rightarrow r_{1k}, r_{2i} \rightarrow r_{2k}, \cdots, r_{ni} \rightarrow r_{nk}]$$

We find $U^{\dagger}(x_i, x_k) = \bigwedge_j [1]$ if $i = k$,

$\bigwedge_j [0]$ if $i \neq k$.

Thus $U^{\dagger} = I$ ----(1.5)

By (1.3) and (1.5), $U^{\dagger} = I \geq U_{\dagger}$.

Remark 3.2

If $R \le T$,then we always not have $U_\dagger \le U^\dagger$. This means $U_\dagger$ and $U^\dagger$ can not comparable or $U_\dagger > U^\dagger$ .If we take $U_\circ = U_\dagger \wedge U^\dagger$,then $U_\circ$ and $U^\dagger$ are solutions of $R \circ U = T$. But $U_\dagger$ may be solution or not. Specially, $R \circ U = R$ is called eigen fuzzy relation equation.

4. Examples

Example 4.1

In case of $R < T$,

$$R = \begin{pmatrix} 0 & 0.1 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.8 & 0.4 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0.1 & 0.2 & 0.6 \\ 0.4 & 0.5 & 0.8 \\ 0.9 & 1 & 0.9 \end{pmatrix}$$

$$U_\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \varnothing .$$

Example 4.2

In case of $R = T$,

$$R = \begin{pmatrix} 0 & 0.1 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.8 & 0.4 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0.1 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.8 & 0.4 & 0 \end{pmatrix}$$

$$U_\dagger = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } U_\dagger \le U^\dagger = I .$$

References

1.E.Sanchez, Resolution of composite fuzzy relation equations, Inform. and Control 30(1976) 38-48

2. J.Drewniak, Equation in classes of fuzzy relations, Fuzzy Sets and Systems 75 (1995) 215-228