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Existence and Uniqueness of Solutions of Fuzzy Integro-Differential Equation with Nonlocal Condition

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Abstract : We will prove the existence and uniqueness theorem of solutions to the nonlocal fuzzy integro-differential equations using Contraction mapping principle.

1. Introduction

The initial value problem

$$x'(t)=f(t,x(t)),\ \ x(0)=x_0$$

has a solution provided ubs $f:I(\subset R)\times X\to X$(Banach space) is continuous and satisfies a Lipsch-itz condition in [3]. The definitions given in this paper generalizes that of Aumann[1] for set-valued mappings. Kaleva [4] discussed the properties of differentiable fuzzy set-valued mappings and gave the existence and uniqueness theorem for a solution of the fuzzy differential equations $f:I(\subset R)\times X\to X$ when f satisfies the Lipschitz condition and he studied the Cauchy Problem of fuzzy differential equations. Byszewski[2] investigated the existence and uniqueness of mild, strong, and classical solutions of a nonlocal Cauchy problem for a semilinear evolution equation. Park and Han([6,7,8]) studied existence of approximate solution for fuzzy differential or integral equation, respectively and investigated the existence and uniqueness of solutions fo r nonlocal fuzzy differential equations. Also, Subrahmanyam and Sudarsanam[10] studied existence theorems for fuzzy Volterra integral equations. In this paper, we prove the existence and uniqueness theorem of solutions to the nonlocal fuzzy integro-differential equations

$$x'(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds),$$
$$x(0) - g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0,$$

where $0 < t_1 < t_2 < \cdots < t_p \leq a$, $f : I \times E^n \times E^n \to E^n$, $k : I^2 \times E^n \to E^n$(the space of all fuzzy sets of R^n with metric $D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$ are levelwise continuous functions which satisfies a generalized Lipschitz condition and also $g : I^p \times E^n \to E^n$ satisfies a generalized Lipschitz condition and $I = [0, a]$. The symbol $g(t_1, t_2, \cdots, t_p, x(\cdot))$ is used in the sence that in the place of $\cdot$ we can substitute only elements of the set $\{t_1, t_2, \cdots, t_p\}$. We denote $I = [0, a]$, a is a positive number. For example $g(t_1, t_2, \cdots, t_p, x(\cdot))$ can be defined by the formula

$$g(t_1, t_2, \cdots, t_p, x(\cdot)) = c_1 x(t_1) + \cdots + c_p x(t_p),$$

where $c_i\ (i = 1, 2, \cdots, p)$ are given constants.

2. Preliminaries

Let $P_K(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n . The distance between A and B is defined by the Hausdorff metric

$$d(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|,\ \sup_{b \in B} \inf_{a \in A} |a - b|\},$$

where $|\cdot|$ denotes the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n), d)$ becom-es a metric space.

Theorem 2.1 [9] The metric space $(P_K(R^n), d)$ is complete and separable.

Let $T = [c, d] \subset R$ be a compact interval and denote

$$E^n = \{u : R^n \to [0, 1] \mid u \text{ satisfies (i)} - \text{(iv) below}\}$$

where

(i) u is normal i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$denote $[u]^\alpha = \{x \in R^n \mid u(x) \geq \alpha\}$, then from (i)-(iv) it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \leq \alpha \leq 1$.

If $h: R^n \times R^n \to R^n$ is any function; then, according to Zadeh's extension principle, we can extend h to $E^n \times E^n \to E^n$ by the equation

$$h(u, v)(z) = \sup_{z=h(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[h(u,v)]^{\alpha} = h([u]^{\alpha}, [v]^{\alpha})$ for all $u, v \in E^n$, $0 \le \alpha \le 1$ and h is continuous. Especially for addition and scalar multiplication, we have

$$[u+v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}, \ [ku]^{\alpha} = k[u]^{\alpha},$$

where $u, v \in E^n$, $k \in R$, $0 \le \alpha \le 1$.

Theorem 2.2 [5] If $u \in E^n$, then

(i) $[u]^{\alpha} \in P_K(R^n)$ for all $0 \le \alpha \le 1$,

(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for $0 \le \alpha_1 \le \alpha_2 \le 1$,

(iii) If $\{\alpha_k\} \subset [0,1]$ is a nondecreasing sequence converging to $\alpha > 0$, then

$$[u]^{\alpha} = \bigcap_{k \ge 1} [u]^{\alpha_k}.$$

Conversely, if $\{A^{\alpha} \mid 0 \le \alpha \le 1\}$ is a family of subsets of R^n satisfying (i)-(iii), then there exists a $u \in E^n$ such that $[u]^{\alpha} = A^{\alpha}$ for $0 < \alpha \le 1$ and $[u]^0 = \overline{\bigcup_{0 < \alpha \le 1} A^{\alpha}} \subset A^0$.

Define $D: E^n \times E^n \to R^+ \cup \{0\}$ by the equation

$$D(u, v) = \sup_{0 \le \alpha \le 1} d([u]^{\alpha}, [v]^{\alpha}),$$

where d is the Hausdorff metric defined in $P_K(R^n)$.

The following definitions and theorems are given in [4].

Definition 2.1 A mapping $F: T \to E^n$ is *strongly measurable* if for all $\alpha \in [0,1]$ the set-valued mapping $F_{\alpha}: T \to P_K(R^n)$ defined by $F_{\alpha}(t) = [F(t)]^{\alpha}$ is Lebesque measurable, when $P_K(R^n)$ is endowed with the topology generated by the Hausdorff metric d and T is a subinterval of real number R.

Definition 2.2 A mapping $F: T \to E^n$ is called *levelwise continuous* at $t_0 \in T$ if the set-valued mapping $F_{\alpha}(t) = [F(t)]^{\alpha}$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0,1]$.

A mapping $F: T \to E^n$ is called *integrably bounded* if there exists an integrable function k such that $|x| \le k(t)$ for all $x \in F_0(t)$.

Definition 2.3 Let $F: T \to E^n$. The integral of F over T, denoted by $\int_T F(t)dt$ or $\int_c^d F(t)dt$, is defined levelwise by the equation

$$\left[\int_T F(t)dt\right]^{\alpha} = \int_T F_{\alpha}(t)\,dt$$
$$= \left\{\int_T f(t) \mid f: T \to R^n \text{ is a measurable selection for } F_{\alpha}\right\}$$

for all $0 < \alpha \le 1$.

Also, a strongly measurable and integrably bounded mapping $F: T \to E^n$ is said to be *integrable* over T if $\int_T F(t)dt \in E^n$

Theorem 2.3 If $F: T \to E^n$ is strongly measurable and integrably bounded, then F is integrable.

It is known that $\left[\int_T F(t)dt\right]^0 = \int_T F_0(t)\,dt$ (see [4]).

Theorem 2.4 Let $F, G: T \to E^n$ be integrable and $\lambda \in R$. Then

(i) $\int_T (F(t) + G(t))dt = \int_T F(t)dt + \int_T G(t)dt,$

(ii) $\int_T \lambda F(t)dt = \lambda \int_T F(t)dt,$

(iii) $D(F, G)$ is integrable,

(iv) $D\left(\int_T F(t)dt, \int_T G(t)dt\right) \le \int_T D(F, G)(t)\,dt.$

Definition 2.4 A mapping $F: T \to E^n$ is *Hukuhara differentiable at* $t_0 \in T$ if for some $h_0 > 0$ the Hukuhara differences

$$F(t_0 + \Delta t) -_h F(t_0), \quad F(t_0) -_h F(t_0 - \Delta t)$$

exist in E^n for all $0 < \Delta t < h_0$ and if there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0+} D\left((F(t_0 + \Delta t) -_h F(t_0))/\Delta t, F'(t_0)\right) = 0$$

and

$$\lim_{\Delta t \to 0+} D\left((F(t_0) -_h F(t_0 - \Delta t))/\Delta t, F'(t_0)\right) = 0.$$

Here $F'(t_0)$ is called the *Hukuhara derivative* of F at t_0.

Definition 2.5 A mapping $F: T \to E^n$ is called differentiable at $t_0 \in T$ if for any $\alpha \in [0, 1]$ the set-valued mapping $F_{\alpha}(\cdot) = [F(\cdot)]^{\alpha}$ are Hukuhara differentiable at point t_0 with Hukuhara derivative $F_{\alpha}'(t_0)$ for each $\alpha \in [0, 1]$.

If $F: T \to E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the *fuzzy derivative* of

$F(t)$ at the point t_0.

Theorem 2.5 Let $F: T \to E^n$ be differentiable and assume that the derivative F' is integrable over T. Then for each $s \in T$, we have

$$F(s) = F(a) + \int_a^s F'(t)dt.$$

Definition 2.6 A mapping $f: T \times E^n \times E^n \to E^n$ is called *levelwise continuous* at a point $(t_0, x_0, y_0) \in T \times E^n \times E^n$ provided that for any fixed $\alpha \in [0, 1]$ and arbitrary $\varepsilon > 0$, there exists a $\delta(\varepsilon, \alpha) > 0$ such that

$$d([f(t, x, y)]^{\alpha}, [f(t_0, x_0, y_0)]^{\alpha}) < \varepsilon,$$

whenever $|t - t_0| < \delta(\varepsilon, \alpha)$, $d([x]^{\alpha}, [x_0]^{\alpha}) < \delta(\varepsilon, \alpha)$ and $d([y]^{\alpha}, [y_0]^{\alpha}) < \delta(\varepsilon, \alpha)$ for all $t \in T$, $x, y \in E^n$.

3. Nonlocal Fuzzy Integro-Differential Equations

Assume that $f: I \times E^n \times E^n \to E^n$, $k: I^2 \times E^n \to E^n$ are levelwise continuous and $g: I^p \times E^n \to E^n$ is a function, where the interval $I = [0, a]$. Consider the fuzzy integrodifferential equation

$$x'(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds), \quad x(0) - g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0, \qquad (3.1)$$

where $x_0 \in E^n$.

Definition 3.1 A mapping $x: I \to E^n$ is a solution to the problem (3.1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) + \int_0^t f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)ds$$

for all $t \in I$.

We shall omit the proofs of following main theorems.

Theorem 3.1 Assume that

(i)A mapping $f: J_0 \to E^n$, $k: J_1 \to E^n$ are levelwise continuous and $g: J_1 \to E^n$ is a function, where $J_0 = I \times \Upsilon \times \Upsilon$, $J_1 = I^2 \times \Upsilon$ and $J_2 = I^p \times \Upsilon$,

$$\Upsilon = \{\varphi \in E^n \,|\, H_1(\varphi, x_0) \le b\}$$

is the space of continuous functions with $H_1(\varphi, \psi) = \sup_{0 \le t \le \xi} D(\varphi(t), \psi(t))$ and b is a positive number.

(ii) For each pairs $(t, x, \varphi), (t, y, \psi) \in J_0$, $(t, s, x), (t, s, y) \in J_1$ and $(t_1, \cdots, t_p, x(\cdot))$,

$(t_1, \cdots, t_p, y(\cdot)) \in J_2$, we have

$$D(f(t,x,\varphi), f(t,y,\psi)) \le L_1[D(x,y) + D(\varphi,\psi)],$$
$$D(k(t,s,x), k(t,s,y)) \le L_2 D(x,y), \tag{3.2}$$
$$D(g(t_1,\cdots,t_p,x(\cdot)), g(t_1,\cdots,t_p,y(\cdot))) \le KD(x,y),$$

where $L_1, L_2, K > 0$ are given constants.

Then there exists a unique solution $x = x(t)$ of (3.1) defined on the interval $[0, \xi]$

where $\xi = \min\{a, \frac{b-N}{M}, \frac{-L_1 + \sqrt{L_1^2 - 4L_1L_2(K-1)}}{2L_1L_2}\}$, $M = D(f(t,x,\varphi), \hat{0})$,

$N = D(g(t_1, \cdots, t_p, x(\cdot)), \hat{0})$, $\hat{0} \in E^n$ such that $\hat{0}(t) = 1$ for $t = 0$ and 0 otherwise and for a ny $(t,x) \in J_0$.

Theorem 3.2 Suppose that f, k, g are the same as in Theorem 3.1. Let $x(t, x_0)$, $y(t, y_0)$ be solutions of equation (3.1) corresponding to x_0, y_0, respectively.

Then there exists a constant $\eta > 0$ such that

$$H_1(x(\cdot, x_0), y(\cdot, y_0)) \le \eta D(x_0, y_0)$$

for any $x_0, y_0 \in E^n$ and $\eta = \frac{1}{1 - K - \xi L_1 - \xi^2 L_1 L_2}$.

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