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Reduced Order H_∞ Controller Synthesis

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Abstract

In this paper, an approach to the reduced order H_∞ controller synthesis is proposed. This approach employs the frequency weighted model reduction whose frequency weights are deduced from the closed-loop system regarding the controller order reduction errors as uncertainties in a plant, while the resultant reduced order H_∞ controller guarantees prescribed H_∞ control performances.

1. Introduction

The standard H_∞ controller has at least the same order as that of the plant to be controlled. When, in particular, a plant has the higher order, the corresponding H_∞ controller may have practically unnecessary orders. However, the lower order controller is desirable in practice, if the resultant performance degradation is kept within an acceptable magnitude. Therefore, reasonable reduced order H_∞ controller synthesis has long been looked for in the field of controller design.

The frequency weighted model reduction is a very useful method in the controller order reduction. This method reduces the order of model on the condition that the reduction error is made small over the frequency range where the gain of frequency weight is high. Today, there are two representative methods for selecting frequency weight used in the problem of H_∞ controller order reduction such that the resultant controller guarantees prescribed H_∞ control performances. These methods regard the reduction error as controller perturbations [1] - [2], and reduce the order of H_∞ controller so that the reduction error is kept within the bound of Youla's free parameter. These methods have advantages such that the resultant reduced order controller certainly guarantees the prescribed H_∞ control performances, while these methods have disadvantages as well.

One, called performance weighted additive reduction, which is the method regarding the reduction error as additive error, needs many steps for making frequency weighted function. The other, called performance weighted coprime factor reduction, which is the method employing left or right coprime factor, is applicable only to the central solution of the H_∞ controllers.

This paper studies H_∞ controller reduction method, employing frequency weighted functions which are made by fewer steps than the performance weighted additive reductions, while the resultant reduced order H_∞ controller guarantees prescribed H_∞ control performances not only for the central solution but for general solution.

This method is realized by regarding the controller order reduction errors as uncertainties in a plant and by utilizing some mathematical properties found in H_∞ norm. An illustrative example is given to verify the consequences.

2. Reduced order H_∞ control problem

2.1 Definition of the reduced order H_∞ control problem

Let a linear time invariant system G be given as

$$G\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{cases} \tag{2.1.1}$$

where

$x(t) \in \Re^n$ is the state vector, $z(t) \in \Re^{p_1}$ the controlled output vector, $y(t) \in \Re^{p_2}$ the measurement output vector, $w(t) \in \Re^{m_1}$, and $w(t) \in L_2$, the disturbance vector, $u(t) \in \Re^{m_2}$ the control input vector,

and the system G satisfies the following standard assumptions;

$$\begin{cases} (C_2, A, B_2) \text{ is stabilizable and detectable} \\ rankD_{12} = m_2,\ rankD_{21} = p_2 \\ rank\begin{bmatrix} A - j\omega I_{n\times n} & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2,\ ^{\forall}\omega \in \Re \quad (2.1.2) \\ rank\begin{bmatrix} A - j\omega I_{n\times n} & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2,\ ^{\forall}\omega \in \Re. \end{cases}$$

The reduced order H$_\infty$ control problem is defined as follows, in the paper.

[Definition].

When an $\hat{n}$-th order H$_\infty$ controller satisfying the H$_\infty$ control performance, which is

$$\underset{K}{G} \in RH_\infty,\ \left\|\underset{K}{G}(z,w)\right\|_\infty < \gamma,\ \gamma > 0 \qquad (2.1.3)$$

is given for the system G in eq. (2.1.1), the reduced order H$_\infty$ control problem is defined as the problem to find an r-th $(0 \le r < \hat{n})$ order controller Kr satisfying the H$_\infty$ control performance given by

$$\underset{Kr}{G} \in RH_\infty,\ \left\|\underset{Kr}{G}(z,w)\right\|_\infty < \gamma,\ \gamma > 0 \qquad (2.1.4)$$

and Kr, a solution of this problem, is defined as the reduced order H$_\infty$ controller, where $\underset{Kr}{G}$ represents lower LFT of G with respect to Kr and the closed-loop system shown in Fig.2.1, $\underset{Kr}{G}(z,w)$ represents the transfer matrix from w to z of $\underset{Kr}{G}$.

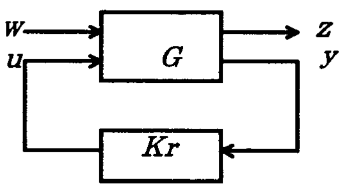


Fig.2.1 Closed-loop system G with H$_\infty$ controller Kr

2.2 An approach to the reduced order H$_\infty$ control problem

Let Kr be assumed to be the resultant reduced-order controller obtained by the application of proposed controller order reduction procedures to the controller K. Then, the approximation error is denoted by $K - Kr$, and Kr could be expressed as $Kr = K - (K - Kr)$.

Furthermore, if the relation

$$\overline{K} = W_u^{-1} K W_y^{-1},\ \ \overline{Kr} = W_u^{-1} Kr W_y^{-1}, \qquad (2.2.2)$$

are defined by the suitably selected nonsingular constant weighted matrices $W_u \in \Re^{m2\times m2}, W_y \in \Re^{p2\times p2}$, and the relation $\overline{K} - \overline{Kr} = W_u^{-1}(K - Kr)W_y^{-1}$ is regarded as the weighted uncertainties, the closed-loop system $\underset{Kr}{G}$ in Fig2.1 can be modified to the closed-loop system $\underset{K}{G_A}^{\overline{K}-\overline{Kr}}$, where $\underset{K}{G_A}^{\overline{K}-\overline{Kr}}$ represents the lower and upper LFT of G with respect to K and $\overline{K} - \overline{Kr}$ and the closed-loop system shown in Fig.2.2.

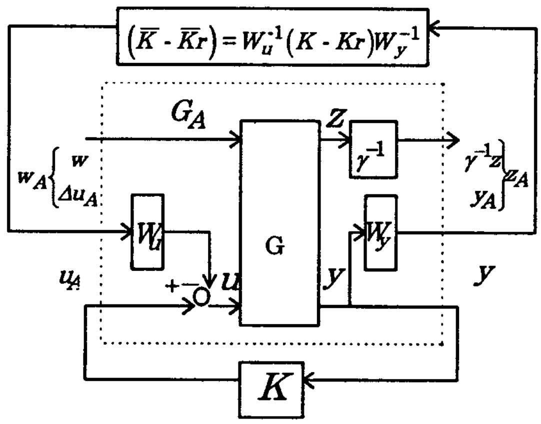


Fig2.2 Closed-loop system $\underset{K}{G_A}^{\overline{K}-\overline{Kr}}$

Thus the following theorem is given.

<Theorem 2.1>

The reduced order controller $Kr\left(= W_u \overline{Kr} W_y\right)$ satisfies

$$\underset{K}{G_A}^{\overline{K}-\overline{Kr}} \in RH_\infty,\ \left\|\underset{K}{G_A}^{\overline{K}-\overline{Kr}}\left(\gamma^{-1}z, w\right)\right\|_\infty < 1, \qquad (2.2.3)$$

then it satisfies

$$\underset{Kr}{G} \in RH_\infty,\ \left\|\underset{Kr}{G}(z,w)\right\|_\infty < \gamma. \qquad (2.2.4)$$

[proof].

The system $G_{\underset{K}{A}}^{\bar{K}-\bar{K}r}$ is equivalent to the system G_{Kr} by the previous statements.

Under the assumption that $\bar{K}-\bar{K}r \in RH_\infty$, we have the relations

$$\|z_A\|_2^2 = \left\|G_{\underset{K}{A}}(z_A,w)w + G_{\underset{K}{A}}(z_A,\Delta u_A)\Delta u_A\right\|_2^2$$
$$= \left\|G_{\underset{K}{A}}(z_A,w)w + G_{\underset{K}{A}}(z_A,\Delta u_A)(\bar{K}-\bar{K}r)y_A\right\|_2^2$$
$$\le \left\|G_{\underset{K}{A}}(z_A,w)w\right\|_2^2 + \left\|G_{\underset{K}{A}}(z_A,\Delta u_A)(\bar{K}-\bar{K}r)y_A\right\|_2^2$$
$$\le \left\|G_{\underset{K}{A}}(z_A,w)\right\|_\infty^2 \|w\|_2^2 + \left\|G_{\underset{K}{A}}(z_A,\Delta u_A)(\bar{K}-\bar{K}r)\right\|_\infty^2 \|y_A\|_2^2 \quad (2.2.5)$$

and

$$\|z_A\|_2^2 = \left\|\begin{bmatrix}\gamma^{-1}z \\ y_A\end{bmatrix}\right\|_2^2 = \left\|\gamma^{-1}z\right\|_2^2 + \|y_A\|_2^2. \quad (2.2.6)$$

Therefore, if the following relations hold

$$\left\|G_{\underset{K}{A}}(z_A,w)\right\|_\infty < 1, \quad \left\|G_{\underset{K}{A}}(z_A,\Delta u_A)(\bar{K}-\bar{K}r)\right\|_\infty < 1, \quad (2.2.7)$$

then we get

$$\left\|\gamma^{-1}z\right\|_2^2 < \|w\|_2^2, \text{ i.e. } \left\|G_{\underset{K}{A}}^{\bar{K}-\bar{K}r}(z,w)\right\|_\infty < \gamma, \quad (2.2.8)$$

and, we also arrive at the following result

$$\left\|G_{\underset{K}{A}}(y_A,\Delta u_A)(\bar{K}-\bar{K}r)\right\|_\infty$$
$$\le \left\|\begin{bmatrix} G_{\underset{K}{A}}(\gamma^{-1}z,\Delta u_A)(\bar{K}-\bar{K}r) \\ G_{\underset{K}{A}}(y_A,\Delta u_A)(\bar{K}-\bar{K}r)\end{bmatrix}\right\|_\infty$$
$$= \left\|G_{\underset{K}{A}}(z_A,\Delta u_A)(\bar{K}-\bar{K}r)\right\|_\infty < 1. \quad (2.2.9)$$

Therefore, we are able to conclude that $G_{\underset{K}{A}}^{\bar{K}-\bar{K}r} \in RH_\infty$ from the small gain theorem.■

Thus the following theorem is given.

< Theorem 2.2 >

For the closed-loop system $G_{\underset{K}{A}}^{\bar{K}-\bar{K}r}$ in Fig. 2.2, if the conditions

$$\begin{cases} (\bar{K}-\bar{K}r) \in RH_\infty, \ \left\|G_{\underset{K}{A}}(z_A,w)\right\|_\infty < 1, \\ \left\|G_{\underset{K}{A}}(z_A,\Delta u_A)(\bar{K}-\bar{K}r)\right\|_\infty < 1 \end{cases} \quad (2.2.10)$$

or

$$\begin{cases} (\bar{K}-\bar{K}r) \in RH_\infty, \ \left\|G_{\underset{K}{A}}(z_A,w)\right\|_\infty < 1, \\ \left\|G_{\underset{K}{A}}(z_A,\Delta u_A)\right\|_\infty < 1, \ \left\|\bar{K}-\bar{K}r\right\|_\infty \le 1 \end{cases} \quad (2.2.11)$$

are satisfied, then

$$G_{Kr} \in RH_\infty, \ \left\|G_{Kr}(z,w)\right\|_\infty < \gamma. \quad (2.2.12)$$

[proof].
This theorem can be straightforwardly verified by the previous statements in this section and theorem 2.1, so the proof is omitted.

From Fig. 2.2, we can recognize that W_y, W_u have the following properties.

1). $\lim_{W_y\to 0}\left\|G_{\underset{K}{A}}(z_A,w)\right\|_\infty < 1$, because

$$\lim_{W_y\to 0}\left\|G_{\underset{K}{A}}(z_A,w)\right\|_\infty = \lim_{W_y\to 0}\left\|G_K(\gamma^{-1}z,w)\right\|_\infty < 1.$$

2). $\lim_{W_u \to 0} \left\| G_{\underset{K}{A}}(z_A, \Delta u_A) \right\|_\infty = 0.$ (2.2.14)

3). W_u has no influence on

$$\left\| G_{\underset{K}{A}}(z_A, \Delta u_A)(\bar{K} - \bar{K}r) \right\|_\infty . \quad (2.2.15)$$

4).

$$\lim_{W_u, W_y \to \infty} \left\| \bar{K} - \bar{K}r \right\|_\infty$$

$$= \lim_{W_u, W_y \to \infty} \left\| W_u^{-1}(K - Kr)W_y^{-1} \right\|_\infty = 0. \quad (2.2.16)$$

Using these four properties, we can establish the following algorithms from theorem 2.2 to solve the reduced order H∞ control problem.

Algorithm.1
Reduction approach with constant matrix weights

STEP 1. Find W_y with the possibly biggest $\|W_y\|_\infty$ satisfying

$$\left\| G_{\underset{K}{A}}(z_A, w) \right\|_\infty < 1.$$

STEP 2. Find W_u with the possibly biggest $\|W_u\|_\infty$ satisfying

$$\left\| G_{\underset{K}{A}}(z_A, \Delta u_A) \right\|_\infty < 1.$$

STEP 3. Find $r(<\hat{n})$-th order controller Kr such that
$\|\bar{K} - \bar{K}r\|_\infty \le 1$, $(\bar{K} - \bar{K}r) \in RH_\infty$.

STEP 4. A reduced order H∞ controller is given by
$Kr = W_u \bar{K}r W_y$.

Algorithm.2 Frequency weighted reduction approach

STEP 1. Find W_y with the possibly biggest $\|W_y\|_\infty$ satisfying

$$\left\| G_{\underset{K}{A}}(z_A, w) \right\|_\infty < 1.$$

STEP 2. Set $W_u = I$, because of eq. (2.2.15).

STEP 3. Find $r(<\hat{n})$-th order controller Kr such that

$$\begin{cases} \left\| G_{\underset{K}{A}}(z_A, \Delta u_A)(\bar{K} - \bar{K}r) \right\|_\infty < 1 \\ (\bar{K} - \bar{K}r) \in RH_\infty \end{cases}$$

STEP 4. A reduced order H∞ controller is given by
$Kr = W_u \bar{K}r W_y$.

3. Example

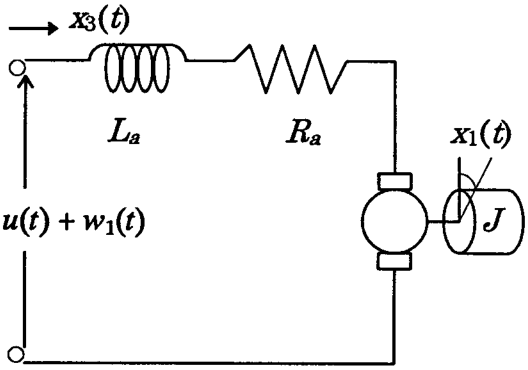


Fig. 3.1 DC Motor

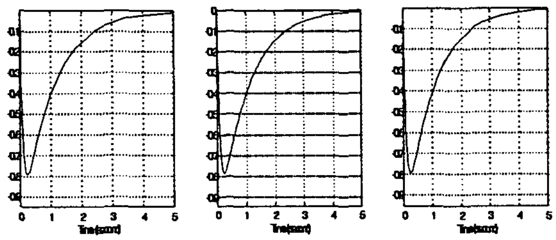
The DC motor shown as in Fig. 3.1 is described by

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -D/J & K_t/J \\ 0 & -K_e/L_a & -R_a/L_a \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/L_a & 0 \end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \quad (3.1)$$

where, $u(t)$ is the armature voltage, $x_1(t)$ the angular, $x_2(t)$ the angular velocity, $x_3(t)$ the armature current, $w_1(t)$ the disturbance, $w_2(t)$ the sensor noise, R_a armature resistance (= 1[Ω]), L_a armature inductance (= 5[mH]), J the moment of inertia (= 0.02[kgm]), K_t torque constant (= 1[Nm/A]), K_e counter emf constant (= 1[Vsec/rad]).

Then, the model of the DC motor can be written numerically as the generalized plant by the appropriate selection of the equation of the equation of controlled output vector $z(t)$;



K $\quad$ $Kr1$ $\quad$ $Kr2$

Fig. 3.4 The time response of $u(t)$

Here, the balanced truncation [6] and the frequency weighted balanced truncation [7] are used respectively to the algorithm 1, and the algorithm 2.

4. Conclusions

We have proposed the reduced order H_∞ controller synthesis employing frequency weights deduced from the closed-loop system which regards the controller order reduction errors as uncertainties in the plant, while the resultant reduced order H_∞ controller guarantees prespecified H_∞ control performances. The verification has also been demonstrated by the illustrative example.

This approach will give the possible foundation for the provision of practical applications.

5. References

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$$G\begin{cases}\dot{x}(t)=\begin{bmatrix}0&1&0\\0&0&50\\0&-200&-200\end{bmatrix}x(t)+\begin{bmatrix}0&0\\0&0\\200&0\end{bmatrix}w(t)+\begin{bmatrix}0\\0\\200\end{bmatrix}u(t)\\ z(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}u(t)\\ y(t)=\begin{bmatrix}1&0&0\end{bmatrix}x(t)+\begin{bmatrix}0&1\end{bmatrix}w(t)\end{cases} \quad (3.2).$$

The 3rd order H_∞ controller K, given by the central solution such that

$$\underset{K_r}{G}\in RH_\infty,\ \left\|\underset{K_r}{G}(z,w)\right\|_\infty<\gamma(=1.7) \quad (3.3)$$

for the (3.2) is

$$K\begin{cases}\dot{x}_c(t)=\begin{bmatrix}-0.9920&1.2048&-0.2629\\-0.3871&10.819&48.816\\-2391.6&-403.32&-290.31\end{bmatrix}x_c(t)+\begin{bmatrix}1.5168\\0.5919\\-0.7597\end{bmatrix}y(t)\\ u(t)=\begin{bmatrix}-11.960&-1.0107&-0.65021\end{bmatrix}x_c(t)\end{cases} \quad (3.4).$$

This controller internally stabilizes the closed-loop $\underset{K}{G}$, and gives $\left\|\underset{K}{G}\right\|_\infty\le 1.1037$.

The reduced order H_∞ controller K_r which guarantees (3.3) can be found using the methods proposed in this paper.

The reduced order H_∞ controller, obtained by the application of algorithm 1 with W_y=0.04, W_u=1.11 to the central solution, is

$$K_{r1}\begin{cases}\dot{x}_{r1}(t)=-12.3009x_{r1}(t)-4.0797y(t)\\ u(t)=2.7219x_{r1}(t)\end{cases} \quad (3.5)$$

This controller has 1st order, and internally stabilizes the closed loop $\underset{K_{r1}}{G}$, and gives $\left\|\underset{K_{r1}}{G}\right\|_\infty\le 1.1023$.

The reduced order H_∞ controller, obtained by the application of algorithm 2 with W_y=0.04 to the central solution, is

$$K_{r2}\begin{cases}\dot{x}_{r2}(t)=-12.0888x_{r2}(t)+0.6145y(t)\\ u(t)=-17.9090x_{r2}(t)\end{cases} \quad (3.6)$$

This controller also has 1st order, and internally stabilizes the closed loop $\underset{K_{r2}}{G}$, and gives $\left\|\underset{K_{r2}}{G}\right\|_\infty\le 1.0931$.

The Gain diagram of the closed-loop system are shown in Fig 3.2. Time responses of the $x_1(t)$, $x_2(t)$, $x_3(t)$ in the resultant closed-loop system are shown in Fig 3.3. Time responses of the control input $u(t)$ in the resultant closed-loop are shown in Fig 3.4.

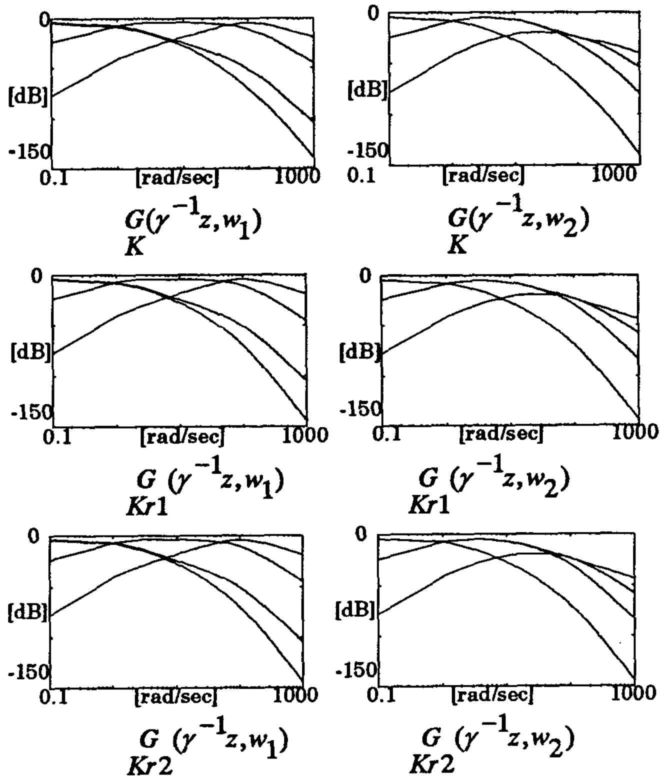

Fig. 3.2 The Gain diagram of the closed loop system

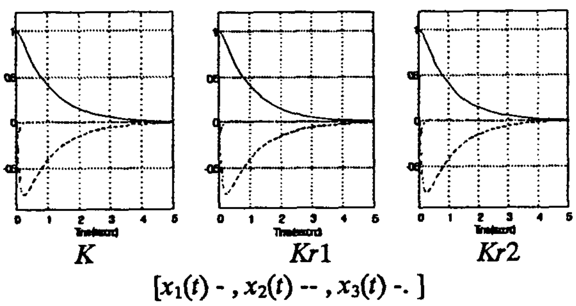

Fig. 3.3 The time response of $x_1(t)$, $x_2(t)$, $x_3(t)$