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Derivation of Recursive Relations in Markov Parameter for the Closed-Loop Identification

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Abstract

This paper presents a closed loop identification algorithm in time domain. This algorithm can be used for identification of unstable system and for model validation of system which is difficult to derive analytical model. In time domain, projection filter, which projects a finite number of input output data of a system into its current space, is used to relate the state space model with a finite difference model. Then recursive relations between the Markov parameters and the ARX model coefficients are derived to identify the system, controller and Kalman filter Markov parameters recursively, which are finally used to identify the system, controller and Kalman filter gains. The NASA LAMSTF is used to validate the algorithms developed.

1. Introduction

System identification is the process of developing or improving mathematical model of physical system based on its input-output data. Achiving high control performance on these system requires accurate model. Such a model can be derived from system identification techniques using experimental data. Recently, methods have been developed to compute the Markov parameters of a linear system from the open loop input output data[1]. By including an observer or Kalman filter in the state space equation, both the open loop plant and associated observer or Kalman filter can be identified and used for further controller design. Further progress was made for identifying an open loop system, observer, and controller from closed loop test data[2]. The great body of literature reveals the importance of the Kalman filter, however, at the same time it reveals the existence of some unsatisfactory features as well. A well known limitation in applying the conventional Kalman filter is its requirement of a priori knowledge about the system state space model and the covariances of process and measurement noises. This information , in practice, is either only partially known or totally unknown. Another limit of the conventional Kalman filter is it can neither adjust itself to trace a changing environment, nor can it correct the error caused by incorrect a priori information. In a sense, the conventional Kalman filter works as an open loop system, because the filter evolves according to present formulas during operations and the estimation error never affects the filter itself. Moreover, after reaching its steady state, the filter "sleeps". That is, no matter how big the estimation error could be due to whatever reasons, the filter just remains unchanged. A phenomenon called filter divergence could happen[3),4),5)]. On the other hand, projection filters are developed to identify stochastic systems from open loop input output data[6]. The approach is primarily based on the relation between the state space model and the AutoRegressive with eXogeneous input (ARX) model, via the projection filter. This paper extends the above development to the identification of an open loop stochastic system operating in a closed loop with or without feedback dynamics.

2. Mathematical Formulation of a System

To explain the relationship between the projection filter and the ARX model of a linear stochastic system, consider a finite-dimensional, linear, discrete, time-invariant stochastic dynamic system represented by a state-space model as

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

$$y_k = Cx_k + Du_k + \nu_k \tag{2}$$

where $x \in R^{n\times 1}, u \in R^{m\times 1}, y \in R^{p\times 1}$ are state, input, and output vectors respectively; $\{w_k\}$ is the process noise, $\{\nu_k\}$ the measurement noise; $[A, B, C, D]$ are the state space parameters. Sequences $\{w_k\}$ and $\{\nu_k\}$ are assumed gaussian, white, zero-mean, stationary with covariances Q and R respectively, and are uncorrelated with each other. The integer k is the sample indicator.

The control input u_k to the plant is the summation of the random excitation signal r_k and the feedback signal $u_{f,k}$. The existing controller can be a full state feedback controller or any dynamic controller. Assuming for the moment, the existing controller is a full state feedback controller with a gain F,

$$u_k = r_k + u_{f,k} \tag{3}$$
$$u_{f,k} = -Fx_k \tag{4}$$

From Eqs. (3) and (4), it is easy to show that

$$\begin{bmatrix} y_k \\ y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-q+1} \end{bmatrix} = \begin{bmatrix} C \\ CA^{-1} \\ CA^{-2} \\ \vdots \\ CA^{-q+1} \end{bmatrix} x_k - \begin{bmatrix} -D & 0 & \cdots & \cdots & 0 \\ 0 & CA^{-1}B-D & \cdots & \cdots & 0 \\ \vdots & CA^{-2}B & CA^{-1}B-D & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & CA^{-q+1}B & \cdots & \cdots & CA^{-1}B-D \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-q+1} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ CA^{-1} & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{-q+1} & \cdots & \cdots & \cdots & CA^{-1} \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ \vdots \\ \vdots \\ w_{k-q+1} \end{bmatrix} + \begin{bmatrix} \nu_k \\ \nu_{k-1} \\ \vdots \\ \vdots \\ \nu_{k-q+1} \end{bmatrix} \tag{5}$$

or in short

$$Y_{q,k} = H_k x_k - G_q U_k - M_q W_{q,k} + V_{q,k} \tag{6}$$

or in a nominal form

$$H_q x_q = Y'_{q,k} + \xi_{q,k} \tag{7}$$

where $Y'_{q,k} = Y_{q,k} + G_q U_k$ and $\xi_{q,k} = M_q W_{q,k} - V_{q,k}$. Note that the unknown x_k is a random variable in this case. Now the overall noise vector $\xi_{q,k}$ is still Gaussian and zero-mean because $W_{q,k}$ and $V_{q,k}$ are Gaussian and zero-mean, but it is correlated with x_k because $W_{q,k}$ is correlated with x_k. Now denote the covariance between x_k and $\xi_{q,k}$ by $P_{x\xi}$ and the auto-covariance of $\xi_{q,k}$ by R_ξ. For Eq. (8), given the mean of the current state $\overline{x_k}$ and its variance P_x, by the theory of random parameters estimation, the optimal estimate of x_k can be obtained by

$$\hat{x}_k = \bar{x}_k + F_q(Y'_{q,k} - \overline{Y'}_{q,k}) \tag{8}$$

where the overbar "-" denotes the expected value,

$$\overline{Y'}_{q,k} = H_q \bar{x}_k \tag{9}$$

and

$$F_q = (P_x H_q^T + P_{x\xi}) \tag{10}$$

The matrix F_q is the projection filter. The optimality is defined by minimum variance of state estimation error. To derive an ARX model, one can form an one step ahead output prediction using the estimated state of the last step, that is,

$$\hat{y}_k = CA\ \hat{x}_{k-1} + CBu_{k-1} + Du_k \tag{11}$$
$$y_k = \hat{y}_k + \varepsilon_k \tag{12}$$
$$\hat{u}_{f,k} = -F\ \hat{x}_k = -F(A\ \hat{x}_{k-1} + Bu_{k-1}) \tag{13}$$
$$u_{f,k} = \hat{u}_{f,k} + \eta_k \tag{14}$$

where the prediction errors ε_k and η_k are the differences between the estimated value and measurement for the output and feedback signal, respectively. Then one has

$$\begin{aligned} \hat{y}_k &= CA\ \hat{x}_{k-1} + CBu_{k-1} + Du_k \\ &= \sum_{i=1}^{q} CAF_{q,i} y_{k-i} + Du_k + (CB - CAF_{q,1}D)u_{k-1} \\ &\quad + \sum_{i=2}^{q} CAF_q G_{q,i} u_{k-i} + CAL\ \bar{x}_{k-1} \end{aligned} \tag{15}$$

where $L = I_n - F_q H_q, I_n$ and $n \times n$ identity matrix and $F_{q,i}$ and $G_{q,i}$ are the i-th partitions of F_q and G_q, respectively, defined as $F_q = [F_{q,1} \vdots F_{q,2} \vdots \cdots \vdots F_{q,q}]$, $G_q = [G_{q,1} \vdots G_{q,2} \vdots \cdots \vdots G_{q,q}]$. Similar to (15), one can derive an ARX model to use one step ahead output prediction for $\hat{u}_{f,k}$

$$\begin{aligned} \hat{u}_{f,k} &= -F\ \hat{x}_k = -F(A\ \hat{x}_{k-1} + Bu_{k-1}) \\ &= -\sum_{i=1}^{q} FAF_{q,i} y_{k-i} - F(AF_q G_{q,1} + B)u_{k-1} \end{aligned}$$

$$- \sum_{i=2}^{q} FAF_q G_{q,i} u_{k-i} - FAL\, \bar{x}_{k-1} \qquad (16)$$

Equations (15) and (16) represent the optimal predictions of y_k and $u_{f,k}$ one can make using q previous input output data. If the prediction is made once and for all, namely, no prediction of the previous state is made, the optimal value assigned to $\bar{x}_k$ is zero. However, if previous state estimation has been made, the optimal choice for $\bar{x}_k$ is the a priori Kalman filter estimate. Note that for a kalman filter

$$\begin{aligned}\bar{x}_{k-1} &= A\,\bar{x}_{k-2} + AK(y_{k-2} - C\,\hat{x}_{k-2} - Du_{k-2}) \\ &\quad + Bu_{k-2} + \cdots \\ &= \sum_{i=1}^{q-1} \bar{A}^{i-1} AKy_{k-1-i} \\ &\quad + \sum_{i=1}^{q-1} \bar{A}^{i-1}(B - AKD)u_{k-1-i} + \bar{A}^q\,\hat{x}_{k-q} \qquad (17)\end{aligned}$$

where $\bar{A} = A(I_n - KC)$ and K is the optimal steady state Kalman filter gain. based on the argument above, one can replace $\bar{x}_{k-1}$ in (15) and (16) by (17) and obtain

$$\begin{aligned}y_k &= \hat{y}_k + \varepsilon_k \\ &= \sum_{i=1}^{q} a_i y_{k-i} + \sum_{i=0}^{q} b_i u_{k-i} + \varepsilon_k + C\,\bar{A}^q\,\hat{x}_{k-q} \qquad (18)\end{aligned}$$

and

$$\begin{aligned}u_{f,k} &= \hat{u}_{f,k} + \eta_k \\ &= \sum_{i=1}^{q} c_i y_{k-i} + \sum_{i=1}^{q} d_i u_{k-i} + \eta_k - FAL\,\bar{A}^q\,\hat{x}_{k-q} \qquad (19)\end{aligned}$$

Note that, although the steady state Kalman filter gain might not be known at the very beginning, it has already existed. This implies that (18) and (19) are valid relations even for the very beginning of the data. In other words, once the value of every input output term in (18) and (19) are known, these equations hold. Note that $\bar{A}$ in (17) is the system matrix of the Kalman filter dynamic equation. Equations (18) and (19) represent the ARX model of a linear system with process and measurement noises. These equations provide the optimal predictions of the output measurement and feedback signal at time k in a sense of minimum state error at time k-1 using q previous input and output data. For a stable filter, the matrix $\bar{A}$ is asymptotically stable. Therefore, the last term of (18) and (19) are negligibly small and can be neglected for a sufficiently large number q. It seems that these ARX models tend to be more accurate as q approaches to infinity. However, if q is too large, one will have overfitting problems[7] due to prediction errors ε_k and η_k which are related to the process and measurement noises which may not be white or zero mean.

3. Derivation of Recursive Relations in Markov Parameter

In this section, the relationships between the system, Kalman filter and controller Markov parameters and the coefficient matrices of the ARX models will be derived. From (18) and (19), if the coefficient matrices of y_{k-j} and u_{k-j} are denoted by a_j, c_j and b_j, d_j respectively, one can derive

$$CA^jB = b_{j+1} + \sum_{i=1}^{j} a_i CA^{j-i}B + a_{j+1}D, \quad j \geq 1 \qquad (20)$$

$$-FA^jB = d_{j+1} + \sum_{i=1}^{j} c_i CA^{j-i}B + c_{j+1}D, \; j \geq 1. \qquad (21)$$

These equations allow one to calculate the system Markov parameter CA^jB, and the controller Markov parameter $-FA^jB$ recursively from the coefficient matrices of an ARX model of order q. Here a proof of (21) is given.

First, by definition, $G_{q,1} = [\,-D^T \; 0^T \; \cdots \; 0^T]^T$, and for $j \geq 2$, one can have

$$\begin{aligned}G_{qj} &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{-1}B - D \\ \vdots \\ CA^{-q+j-1}B \end{bmatrix} \\ &= \begin{bmatrix} C \\ \vdots \\ CA^{-j+2} \\ CA^{-j+1} \\ \vdots \\ CA^{-q+1} \end{bmatrix} A^{j-2}B - \begin{bmatrix} CA^{j-2} \\ \vdots \\ C \\ 0 \\ \vdots \\ 0 \end{bmatrix} B - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D \\ \vdots \\ 0 \end{bmatrix} \\ &= H_q A^{j-2}B - E^{(j-2)}B - D^{(j)} \qquad (22)\end{aligned}$$

where

$$E^{(j-2)} = [\,(CA^{j-2})^T, \; \cdots, \; C^T, \; 0^T, \; \cdots \; 0^T]^T,$$

$$D^{(j)} = [\,0^T \;, \cdots \;, 0^T \;, D^T \;, \cdots \; 0^T],$$

Therefore, with $j \geq 1$

$$\begin{aligned}&CAF_q G_{q,j+1} \\ &= CAF_q H_q A^{j-1}B - CAF_q E^{(j-1)}B - CAF_{q,j+1}D \\ &= CAF_q H_q A^{j-1}B\end{aligned}$$

$$- \sum_{i=1}^{j} CAF_{q,i}CA^{j-i}B - CAF_{q,j+1}D \tag{23}$$

Then using (19), (23), and the relations $\overline{A}+AKC=A$ and $F_qH_q+L=I_n$, one has

$$\begin{aligned}
&-FA^jB \\
&= -FAA^{j-1}B = -FA(F_qH_q+L)A^{j-1}B \\
&\quad -FAF_qH_qA^{j-1}B - FALA^{j-1}B \\
&= -FAF_qH_qA^{j-1}B - FAL(\overline{A}+AKC)A^{j-2}B \\
&= -FAF_qH_qA^{j-1}B - FAL\overline{A}A^{j-2}B - FALAKCA^{j-2}B \\
&= \cdots\cdots \\
&= -FAF_qH_qA^{j-1}B - FAL\,\overline{A}^{j-2}AB \\
&\quad -\sum_{i=1}^{j-2} FAL\,\overline{A}^{i-1}AKCA^{j-i-1}B \\
&= -FAF_qH_qA^{j-1}B - FAL\,\overline{A}^{j-2}(\overline{A}+AKC)B \\
&\quad -\sum_{i=1}^{j-2} FAL\,\overline{A}^{i-1}AKCA^{j-i-1}B \\
&= -FAF_qH_qA^{j-1}B - FAL\,\overline{A}^{j-1}B \\
&\quad -\sum_{i=1}^{j-2} FAL\,\overline{A}^{i-1}AKCA^{j-i-1}B - FAL\,\overline{A}^{j-2}AKCB \\
&= -FAF_qH_qA^{j-1}B - \sum_{i=1}^{j} FAF_{q,i}CA^{j-i}B + FAF_{q,j+1}D \\
&-FAL\,\overline{A}^{j-1}(B-AKD) - \sum_{i=1}^{j} FAF_{q,i}CA^{j-i}B \\
&-\sum_{i=1}^{j-1} FAL\,\overline{A}^{i-1}AKCA^{j-i-1}B - FA(F_{q,j+1}+L\,\overline{A}^{j-1}AK)D \\
&= -FAF_qG_{q,j+1} - FAL\,\overline{A}^{j-1}(B-AKD) \\
&-\sum_{i=1}^{j} FAF_{q,i}CA^{j-i}B - \sum_{i=1}^{j-1} FAL\,\overline{A}^{i-1}AKCA^{j-i-1}B \\
&-FA(F_{q,j+1}+L\,\overline{A}^{j-1}AK)D \\
&= -FA(F_qG_{q,j+1} - L\,\overline{A}^{j-1}(B-AKD)) \\
&\quad -FAF_{q,1}CA^{j-1}B - FA(F_{q,2}+LAK)CA^{j-2}B \\
&\quad -\cdots\cdots - FA(F_qG_{q,j}+L\,\overline{A}^{j-2}AK)CB \\
&\quad -FA(F_{q,j+1}+L\,\overline{A}^{j-1}AK)D \\
&= -FA(F_qG_{q,j+1} - L\,\overline{A}^{j-1}(B-AKD)) \\
&\quad -FAF_{q,1}CA^{j-1}B \\
&\quad -\sum_{i=2}^{j} FA(F_{q,i}+L\,\overline{A}^{i-2}AK)CA^{j-i}B \\
&\quad -FA(F_{q,j+1}+L\,\overline{A}^{j-1}AK)D \\
&= d_{j+1} + \sum_{i=1}^{j} c_iCA^{j-i}B + c_{j+1}D.
\end{aligned} \tag{24}$$

Similar to (20), (21), there are another two equations to obtain the Kalman filter Markov parameter $CA^{j+1}K$ and the Kalman filter/controller Markov parameter $-FA^{j+1}K$ recursively from the coefficient matrices of the ARX models as follows:

$$CA^{j+1}K = a_{j+1} + \sum_{i=1}^{j} a_{j-i+1}CA^iK, \quad j \geq 1 \tag{25}$$

$$-FA^{j+1}K = c_{j+1} + \sum_{i=1}^{j} c_{j-i+1}CA^iK, \quad j \geq 1. \tag{26}$$

Note that $CAK=a_1$ and $-FAK=c_1$. Here a proof of (25) is given and one can derive (26) in a similar way. Using (18) and the relations $\overline{A}+AKC=A$, and $F_qH_q+L=I_n$, one has

$$\begin{aligned}
&CA^{j+1}K \\
&= CA(F_qH_q+L)A^jK \\
&= CAF_qH_qA^jK + CALA^jK \\
&= CA(F_{q,j+1}+F_{q,j}CA^{-j+1}A^jK + F_{q,j-1}CA^{-j+2}A^jK \\
&\quad +\cdots + F_{q,1}CA^jK) + CA(L\overline{A}A^{j-1}K + LAKCA^{j-1}K) \\
&= \cdots\cdots \\
&= CA(F_{q,j+1}+F_{q,j}CAK + F_{q,j-1}CA^2K + \cdots + F_{q,1}CA^jK) \\
&= CAF_{q,j+1} + \sum_{i=1}^{j} CAF_{q,j-i+1}CA^iK + CAL\,\overline{A}^{j-1}AK \\
&\quad + \sum_{i=1}^{j-1} CAL\,\overline{A}^{j-i-1}AKCA^iK \\
&\quad + CA(L\,\overline{A}^{j-1}AK + A\,\overline{A}^{j-2}AKCAK + \cdots \\
&\quad + CAL\overline{A}AKCA^{j-2}K + CALAKCA^{j-1}K) \\
&= CA(F_{q,j+1}+L\,\overline{A}^{j-1}AK) \\
&\quad + \sum_{i=1}^{j} CA(F_{q,j-i+1}+L\,\overline{A}^{j-i+1}AK)CA^iK \\
&= a_{j+1} + \sum_{i=1}^{j} a_{j-i+1}CA^iK.
\end{aligned} \tag{27}$$

To convert the identified ARX model to the state space model, the state space realization is needed. The common realization technique is Singular Value Decomposition (SVD) method. The SVD also provides the function of model reduction by examining the magnitude of the singular values. See reference 8 for details. The first step is to form a Hankel matrix from Markov parameters as follows:

$$H(j-1) = \begin{bmatrix} Y_j & Y_{j+1} & \cdots & Y_{j+\beta} \\ Y_{j+1} & Y_{j+2} & \cdots & Y_{j+\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{j+\gamma} & Y_{j+\gamma+1} & \cdots & Y_{j+\gamma+\beta} \end{bmatrix} \tag{28}$$

where Y_i is the i-th Markov parameter. From the measurement Hankel matrix, the realization uses the SVD of $H(0)$, $H(0)=U\sum V^T$, to identify a k-th order discrete state space model as

$$
\begin{aligned}
A &= \Sigma_k^{-1/2} U_k^T H(1) V_k \Sigma_k^{-1/2} \\
B &= \Sigma_k^{1/2} V_k^T E_r \\
C &= E_m^T U_k \Sigma_k^{1/2}
\end{aligned}
\quad (29)
$$

where matrix Σ_k is the upper left hand $k \times k$ partition of $k \times k$ containing the k largest singular values along the diagonal. Matrices U_k and V_k are obtained from U and V by retaining only the k columns of singular vectors associated with the k singular values. Matrix E_m is a matrix of appropriate dimension having m columns, all zero except that the top $m \times m$ partition is an identity matrix. E_r is defined analogously.

4. Summary of Procedure

In this section, we summarize the procedure for identifying the open loop state space model from the closed loop input output data.

1. Estimate the coefficients of ARX model from input output data by least square method.

2. Calculate Markov parameters recursively from the coefficients of the ARX model.

3. Realize the open loop state space model from the open loop system Markov parameters through Singular Value Decomposition (SVD) method.

4. Calculate the open loop Kalman filter gain from the open loop Kalman filter Markov parameters.

In order to demonstrate the feasibility of the method developed here, numerical simulation and test data from LAMSTF (Large Angle Magnetic Suspension Test Facility) are used. The simulated closed-loop step responses from the identified model and analytical model are compared with the test data. Figure shows the responses of the testing data, analytical model and identified model for a yaw step command.

5. Concluding Remarks

In this paper, the main contribution is derivation of the recursive relations in Kalman filter and controller Markov parameters from the ARX model coefficients which are function of the closed loop input output data. Also the projection filter is developed to identify an open loop stochastic system operating under closed loop condition. This algorithm can be applied to identification of the open loop system which has difficulties to get accurate analytical model.

6. References

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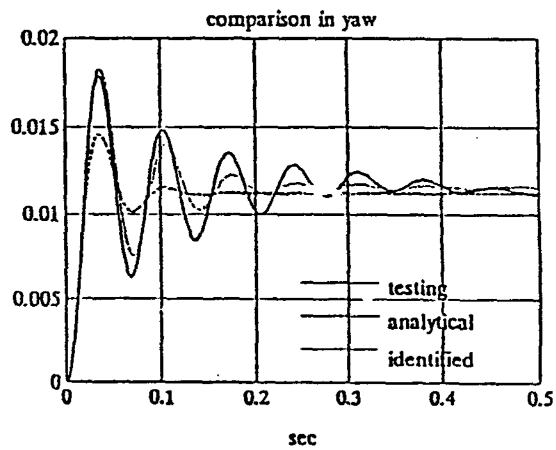

Figure The step response from testing, analytical and identified model for the yaw step.