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LQ Regulator of Systems with Multiple Time-Delays by Memoryless Feedback

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Abstract

A method to construct a memoryless feedback law for systems with multiple time-delays in the states is proposed. As a plant model, a differential-difference equation with multiple delayed terms is introduced. A stabilizability condition by memoryless feedback is presented. A feedback gain is calculated with a solution of a finite dimensional Riccati equation. It is shown that the resulting closed loop system is asymptotically stable, and moreover, it is a linear quadratic regulator for some cost functional. An alternative stabilizability condition which is easier to check is given.

1. Introduction

Systems with time-delay in the states belong to a class of infinite dimensional systems, so it is difficult to implement the state feedback for them. To avoid this problem, the memoryless feedback was proposed. Assuming this type of feedback, some stabilizability conditions have been derived. Ones via the solution of a Liapunov or a Riccati equation were given by Nazaroff-Hewer [1], Kwon-Pearson [2] and Feliachi-Thowsen [3]. Ikeda-Ohta [4] showed a condition which can be checked directly from the plant parameters. This condition was extended by Ryo-Ikeda-Kitamura [5], Yasuda-Hirai [6], Miguchi-Ikeda [7], Amemiya [8] and Akazawa-Amemiya-Tokumaru [9]. An alternative condition was given by Furukawa-Shimemura [10]. Memoryless stabilization of systems with uncertain parameters was studied by Trinh-Aldeen [11], Choi-Chung [12] and Wu-Mizukami [13]. Disturbance rejection using the memoryless feedback was investigated by Lee-Kim-Kwon [14], Choi-Chung [15] and He-Wang-Lee [16].

Recently, Shimemura and the present author proposed a new method to construct a memoryless feedback law for systems with time-delay in the states [17]. The merits of this method are the followings;

(i) the resulting closed loop system is an LQ regulator for some cost functional, and

(ii) this regulator can be obtained without solving any infinite dimensional Riccati equation.

Furthermore, this regulator is assured to have good robustness properties as well as a finite dimensional LQ regulator; it has

(a) infinite gain margin,

(b) at least 50 percent gain reduction tolerance,

(c) at least $\pm 60°$ phase margin, and

(d) robust stability against a class of nonlinear perturbations

[18].

In this paper, the above method is extended for systems with multiple time-delays in the states. First, a differential-difference equation is introduced as a plant model. Second, a Riccati-like finite dimensional matrix equation is presented. With its symmetric positive definite solution, a memoryless feedback law is constructed. Then it is shown that the resulting closed loop system is asymptotically stable, and it minimizes some cost functional, that is, it belongs to a class of LQ regulators. Lastly, an alternative stabilizability condition which is easier to check is given.

2. Formulation

Let us consider a plant described by

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + Bu(t) \qquad (1)$$

$$x(0) = \phi(0)$$

$$x(\theta) = \phi_\theta \quad -h \le \theta \le 0$$

where $x(t) \in R^n$, $u(t) \in R^m$, $A_0 \in R^{n\times n}$, $A_i \in R^{n\times n}$ $(i = 1, \cdots, k)$, $B \in R^{n\times m}$, $0 < h_1 < \cdots < h_k$, and the initial condition is $\{\phi(0), \phi_\theta\}$. The plant parameters are assumed to satisfy the following condition:

Condition 1

There exist

$$\alpha \in R, \alpha > 0$$

$$Q \in R^{n\times n}, Q \ge 0$$

$$R \in R^{m\times m}, R > 0$$

$$\delta_i \in R, \delta_i > 0 \quad (i = 1, ..., k)$$

$$P \in R^{n\times n}, P > 0$$

such that

$$PA_0 + A_0^T P + \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} A_i^T P A_i + 2\left(\sum_{i=1}^{k} \delta_i + \alpha\right) P - PBR^{-1}B^T P + Q = 0. \qquad (2)$$

□

Using these P and R, a memoryless feedback law

$$u(t) = -R^{-1}B^T P x(t) \qquad (3)$$

is constructed, and the closed loop system

$$\frac{dx(t)}{dt} = \left(A_0 - BR^{-1}B^T P\right) x(t) + \sum_{i=1}^{k} A_i x(t-h_i) \qquad (4)$$

is obtained.

In this paper, it is shown that the closed loop system (4) is asymptotically stable, and the memoryless feedback law (3) is the optimal control for the plant (1) minimizing the cost functional:

$$J = \int_0^\infty q(x,u)dt \qquad (5)$$

$$q(x,u) = x^T(t)\tilde{Q}x(t) + u^T(t)Ru(t) + \sum_{i=1}^{k} \|P^{1/2}\{\frac{\sqrt{2\delta_i}}{e^{\alpha h_i}}x(t) - \frac{e^{\alpha h_i}}{\sqrt{2\delta_i}}A_i x(t-h_i)\}\|^2 \ge 0 \qquad (6)$$

$$\tilde{Q} = Q + 2\alpha P + 2\sum_{i=1}^{k} \delta_i (1 - e^{-2\alpha h_i})P > 0 \qquad (7)$$

where $\|\cdot\|$ denotes the Eucledian norm.

3. Linear Quadratic Regulator

In this section, two theorems are given. The first one is to show the stability of the closed loop system (4). The second one is to show the optimality of the memoryless feedback (3).

Theorem 1

Under Condition 1, the closed loop system (4) is asymptotically stable.

Proof:

Let us consider the function

$$V(t) = x^T(t)Px(t) + \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} \int_{-h_i}^{0} x^T(t+\theta)A_i^T PA_i x(t+\theta)d\theta. \quad (8)$$

Differentiating $V(t)$ by the time t, we have

$$\begin{aligned}\frac{dV(t)}{dt} &= \left\{\frac{dx(t)}{dt}\right\}^T Px(t) + x^T(t)P\frac{dx(t)}{dt} \\ &+ \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} \int_{-h_i}^{0} \left\{\frac{\partial x(t+\theta)}{\partial t}\right\}^T \cdot A_i^T PA_i x(t+\theta)d\theta \\ &+ \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} \int_{-h_i}^{0} x^T(t+\theta) \cdot A_i^T PA_i \left\{\frac{\partial x(t+\theta)}{\partial t}\right\} d\theta.\end{aligned} \quad (9)$$

Noting that

$$\frac{\partial x(t+\theta)}{\partial t} = \frac{\partial x(t+\theta)}{\partial \theta}, \quad (10)$$

we have

$$\begin{aligned}&\int_{-h_i}^{0} x^T(t+\theta)A_i^T PA_i \left\{\frac{\partial x(t+\theta)}{\partial t}\right\} d\theta \\ &= x^T(t)A_i^T PA_i x(t) - x^T(t-h_i)A_i^T PA_i x(t-h_i) \\ &- \int_{-h_i}^{0} \left\{\frac{\partial x(t+\theta)}{\partial t}\right\}^T A_i^T PA_i x(t+\theta)d\theta. \end{aligned} \quad (11)$$

Using this equation to Equation (9),

$$\begin{aligned}\frac{dV(t)}{dt} &= \left\{\frac{dx(t)}{dt}\right\}^T Px(t) + x^T(t)P\frac{dx(t)}{dt} \\ &+ \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} x^T(t)A_i^T PA_i x(t) \\ &- \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} x^T(t-h_i)A_i^T PA_i x(t-h_i)\end{aligned} \quad (12)$$

is obtained. Along the solution of Equation (1),

$$\begin{aligned}\frac{dV(t)}{dt} &= \{A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + Bu(t)\}^T Px(t) \\ &+ x^T(t)P\{A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + Bu(t)\} \\ &+ \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} x^T(t)A_i^T PA_i x(t) \\ &- \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} x^T(t-h_i)A_i^T PA_i x(t-h_i) \\ &= x^T(t)\{A_0^T P + PA_0 \\ &+ \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} A_i^T PA_i + \sum_{i=1}^{k} \frac{2\delta_i}{e^{2\alpha h_i}} P\}x(t) \\ &- \sum_{i=1}^{k} \|P^{1/2}\{\frac{\sqrt{2\delta_i}}{e^{\alpha h_i}} x(t) - \frac{e^{\alpha h_i}}{\sqrt{2\delta_i}} A_i x(t-h_i)\}\|^2 \\ &+ u^T(t)B^T Px(t) + x^T(t)PBu(t)\end{aligned} \quad (13)$$

holds. With the Riccati Equation (2), we obtain

$$\begin{aligned}\frac{dV(t)}{dt} &= x^T(t)\{\sum_{i=1}^{k} \frac{2\delta_i}{e^{2\alpha h_i}} P \\ &-2\left(\sum_{i=1}^{k} \delta_i + \alpha\right) P - Q + PBR^{-1}B^T P\}x(t) \\ &- \sum_{i=1}^{k} \|P^{1/2}\{\frac{\sqrt{2\delta_i}}{e^{\alpha h_i}} x(t) - \frac{e^{\alpha h_i}}{\sqrt{2\delta_i}} A_i x(t-h_i)\}\|^2 \\ &+ u^T(t)B^T Px(t) + x^T(t)PBu(t) \\ &= -x^T(t)\tilde{Q}x(t) - u^T(t)Ru(t) \\ &- \sum_{i=1}^{k} \|P^{1/2}\{\frac{\sqrt{2\delta_i}}{e^{\alpha h_i}} x(t) - \frac{e^{\alpha h_i}}{\sqrt{2\delta_i}} A_i x(t-h_i)\}\|^2 \\ &+ \|R^{1/2}\{u(t) + R^{-1}B^T Px(t)\}\|^2.\end{aligned} \quad (14)$$

Substituting this equation to the identity

$$V(0) = V(\tau) - \int_0^{\tau} \frac{dV(t)}{dt} dt, \tag{15}$$

we obtain

$$\begin{aligned} V(0) & \\ = \; & V(\tau) + \int_0^{\tau} \left\{ x^T(t)\tilde{Q}x(t) + u^T(t)Ru(t) \right\} dt \\ + \; & \int_0^{\tau} \sum_{i=1}^{k} \| P^{1/2} \{ \frac{\sqrt{2\delta_i}}{e^{\alpha h_i}} x(t) \\ & - \frac{e^{\alpha h_i}}{\sqrt{2\delta_i}} A_i x(t-h_i) \} \|^2 dt \\ - \; & \int_0^{\tau} \| R^{1/2} \{ u(t) + R^{-1} B^T P x(t) \} \|^2 dt. \end{aligned} \tag{16}$$

If the feedback law (3) is used,

$$\begin{aligned} V(0) & \\ = \; & V(\tau) + \int_0^{\tau} \left\{ x^T(t)\tilde{Q}x(t) + u^T(t)Ru(t) \right\} dt \\ + \; & \int_0^{\tau} \sum_{i=1}^{k} \| P^{1/2} \{ \frac{\sqrt{2\delta_i}}{e^{\alpha h_i}} x(t) \\ & - \frac{e^{\alpha h_i}}{\sqrt{2\delta_i}} A_i x(t-h_i) \} \|^2 dt \\ \geq \; & \int_0^{\tau} x^T(t) \tilde{Q} x(t) dt \end{aligned} \tag{17}$$

holds. The right hand side of this inequality is monotone increasing according to τ, whereas the left hand side is a constant. As $\tau \to \infty$, this integral exists. With $\tilde{Q} > 0$, we conclude that the closed loop system (4) is asymptotically stable, that is,

$$\lim_{t \to \infty} x(t) = 0. \tag{18}$$

□

Theorem 2

Under Condition 1, the memoryless feedback (3) is the optimal control minimizing the cost functional (5).

Proof:

As $\tau \to \infty$, Equation (16) gives

$$\begin{aligned} V(0) & \\ = \; & V(\infty) + \int_0^{\infty} q(x,u) dt \\ - \; & \int_0^{\infty} \| R^{1/2} \{ u(t) + R^{-1} B^T P x(t) \} \|^2 dt. \end{aligned} \tag{19}$$

From Equation (18), we know $V(\infty) = 0$. So, substituting Equation (5), we know

$$\begin{aligned} J = \; & V(0) \\ & + \int_0^{\infty} \| R^{1/2} \{ u(t) + R^{-1} B^T P x(t) \} \|^2 dt. \end{aligned} \tag{20}$$

As $V(0)$ is a constant, this implies that the control (3) is the optimal one in the class of stabilizing controls.

□

Thus we know that the closed loop system (4) belongs to a class of LQ regulators.

4. Stabilizability Condition

Condition 1 has a rather implicit form, and it is difficult to check whether a given plant satisfies it or not directly from its parameters. Now let us consider the following condition:

Condition 2

$$(A_0, B) : controllable.$$

$$rank B = m.$$

There exist $L \in R^{nk \times l}$, $C_0 \in R^{l \times n}$, $\alpha > 0$ and $\delta > 0$ such that

$$(C_0, A_0) : observable,$$

$$rank C_0 = l,$$

$$rank \begin{bmatrix} sI - A_0 & B \\ C_0 & 0 \end{bmatrix} = n + l$$

$$\forall s \in C_\alpha \quad C_\alpha = \{ s \in C; Re[s] > -\delta - \alpha \}$$

and

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} = LC_0.$$

□

Then the following theorem holds;

Theorem 3

Condition 2 is a sufficient condition for Condition 1.

Proof:

Divide L as

$$L = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_k \end{bmatrix} \quad L_i \in R^{n\times l} \quad (i = 1, 2, \cdots, k), \tag{21}$$

and we can write

$$A_i = L_i C_0 \quad (i = 1, 2, \cdots, k). \tag{22}$$

Let us set

$$\delta_i = \frac{\delta}{k} \quad (i = 1, 2, \cdots, k). \tag{23}$$

If the plant (1) satisfies Condition 2, then the finite dimensional Riccati equation

$$\begin{aligned} &P_\sigma\{A_0 + (\alpha+\delta)I\} + \{A_0 + (\alpha+\delta)I\}^T P_\sigma \\ &-\sigma P_\sigma BB^T P_\sigma + C_0^T C_0 = 0 \end{aligned} \tag{24}$$

$$\sigma \in R, \sigma > 0$$

has a symmetric positive definite solution $P_\sigma \in R^{n\times n}$ and it has a special property that $P_\sigma \to 0$ as $\sigma \to \infty$ [10, Lemma 6]. So for σ large enough, there exists P_σ such that

$$\sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} L_i^T P_\sigma L_i < I. \tag{25}$$

We rewrite this P_σ as P. P satisfies

$$\begin{aligned} &\sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} C_0^T L_i^T P L_i C_0 \\ &= \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} A_i^T P A_i \\ &\le C_0^T C_0. \end{aligned} \tag{26}$$

Let us select

$$Q = C_0^T C_0 - \sum_{i=1}^{k} \frac{e^{2\alpha h_i}}{2\delta_i} A_i^T P A_i > 0 \tag{27}$$

$$R = \frac{1}{\sigma} I > 0. \tag{28}$$

Substituting these parameters into Equation (24) gives Equation (2). This means that α, Q, R, δ_i $(i = 1, 2, \cdots, k)$ and P satisfy Equation (2).

□

From this proof, we know how to find parameters α, Q, R, δ_i $(i = 1, 2, \cdots, k)$ and P which satisfy Equation (2). The algorithm is as follows:

1. Select α and δ satisfying Condition 2.
2. Select $\sigma > 0$.
3. Calculate the solution P_σ of Equation (24).
4. Check whether Inequality (25) is satisfied. If not, return to step (ii) and increase σ.
5. Let $P = P_\sigma$ and calculate Q, R, δ_i $(i = 1, 2, \cdots, k)$ from α, δ, σ, P.

The integrand (6) of the cost functional (5) is determined with these parameters. So this algorithm can be regarded as a method to select weighting parameters so as to make the optimal control a memoryless state feedback.

5. Conclusion

A method to construct a memoryless feedback law for systems with multiple time-delays in the states was proposed. A stabilizability condition by memoryless feedback was presented.

It was shown that the resulting closed loop system was asymptotically stable, and moreover, it was a linear quadratic regulator for some cost functional. An alternative stabilizability condition which was easier to check was given.

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