Proceedings of the 13th
KACC, October 1998

Regional Identifiability of Spatially-Varying Parameters in Distributed Parameter Systems of Hyperbolic Type

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Abstract

This paper studies the regional identifiability of spatially-varying parameters in distributed parameter systems of hyperbolic type. Let Ω be a bounded domain of $\mathbf{R}^n$ and let Ω_0 be a subregion of the closed domain $\overline{\Omega}$. The distributed parameter systems having unknown parameters defined on $\overline{\Omega}$ are described by the second order evolution equations in the Hilbert space $L^2(\Omega)$ and the observations are made on the subregion $\Omega_0 \subset \overline{\Omega}$. The regional identifiability is formulated as the uniqueness of parameters by the identity of solutions on the subregion. Several regional identifiability results of the spatially-varying parameters of hyperbolic distributed parameter systems are established by means of the Riesz spectral representations.

1. Regional Identifiability for Evolution Equations

Recently, the new concepts of regional controllability and observability are introduced and their properties are studied by El Jai *et al* [2,3]. The regional concepts reflect the importance in practical applications, which often require the controllability and/or observability in a prescribed subregion of the entire region. Such regional concepts should be studied for the identifiability. In many practical identification problems of unknown parameters in distributed parameter systems, the unknown parameters are only required to be identified in a desired subregion of the entire region. The identification in a subregion has been studied in the output least squares formulation. However the research on the identifiability of parameters in the subregion, called the regional identifiability, is very few (Kunisch and Nakagiri [6]). In this paper, the concept of regional identifiability is formulated in the framework of abstract second order evolution equations, and is studied for several classes of hyperbolic distributed parameter systems by means of the Riesz spectral representations and cosine families.

Let Ω be a bounded domain of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and let Ω_0 be a subregion of the closure $\overline{\Omega}$. Let Q be a space of parameters q defined on the closed domain $\overline{\Omega}$. The underlying Hilbert space $L^2(\Omega)$ is denoted simply by H and the inner product and norm are denoted by $(\cdot,\cdot)$ and $|\cdot|$. For each $q \in Q$ there exists a bilinear form $a(q;\phi,\varphi)$ defined on the products of Hilbert spaces $V(q) \times V(q)$ such that the pair $(V(q), H)$ is a Gelfand triple space with a notation, $V(q) \hookrightarrow H \equiv H' \hookrightarrow V(q)'$. This notation means that an embedding $V(q) \subset H$ is continuous and $V(q)$ is dense in H, so that the embedding $H \subset V(q)'$ is also continuous and the identified $H \equiv H'$ is dense in $V(q)'$. We denote the norm of $V(q)$ by $\|\cdot\|_{V(q)}$. The bilinear form $a(q;\phi,\varphi)$ is assumed to satisfy

$$\begin{aligned} &\text{there exists } c_q > 0 \text{ such that} \\ &|a(q;\phi,\varphi)| \le c_q \|\phi\|_{V(q)} \|\varphi\|_{V(q)} \quad \forall \phi,\psi \in V(q); \quad (1) \\ &\text{there exist } \alpha_q > 0 \text{ and } \lambda_q \in (-\infty,+\infty) \text{ such that} \\ &a(q;\phi,\phi) + \lambda_q|\phi|^2 \ge \alpha_q \|\phi\|^2_{V(q)} \quad \forall \phi \in V. \quad (2) \end{aligned}$$

Then we can define the operator $A(q) \in \mathcal{L}(V,V')$ deduced by the relation

$$a(q;\phi,\varphi) = \langle A(q)\phi,\varphi\rangle_{V(q)',V(q)} \quad \forall \phi,\varphi \in V(q), \qquad (3)$$

where $\langle\cdot,\cdot\rangle_{V(q)',V(q)}$ denotes the duality pairing between $V(q)'$ and $V(q)$. Consider the system described by the second order evolution equation in the Gelfand triple space $(V(q), H)$:

$$\left\{ \begin{array}{l} \dfrac{d^2u}{dt^2} + A(q)u = f(t;q), \quad t > 0 \\ u(0) = a(q), \ \frac{du}{dt}(0) = b(q), \end{array} \right. \qquad (4)$$

where $a(q)$, $b(q)$ and $f(t;q)$ are initial values and a forcing function depending on q such that $a(q) \in V(q)$, $b(q) \in H$ and $f(t;q) \in L^2(0,T;H)$ for all $T > 0$. Then by Lions and Magenes [8] and Dautray and Lions [4], the weak (or variational) solution $u = u(t;q)$ to (1) exists uniquely which satisfies

$$u \in L^2(0,T;V(q)),\ u' \in L^2(0,T;H),\ u'' \in L^2(0,T;V(q)'). \tag{5}$$

This solution $u(t;q)$ is called the state of (4).

Let Ω_0 be an observable subregion of $\overline{\Omega}$ of positive Lebesgue measure. The observation of the state $u(t;q)$ of (4) is made by

$$u(t;q)|_{\Omega_0} = \chi_{\Omega_0} u(t;q), \tag{6}$$

where χ_{Ω_0} is the characteristic operator from $L^2(\Omega)$ to $L^2(\Omega_0)$ (cf. El Jai *et al* [3]). Note that $V(q) \subset H = L^2(\Omega)$.

The regional identifiability problem of unknown parameter q in Ω_0 is formulated as follows. Let $q^m \in Q$ be a known model parameter. The model evolution equation system in the Gelfand triple space $(V(q^m), H)$ is given by

$$\begin{cases} \dfrac{d^2u}{dt^2} + A(q^m)u = f(t;q^m), & t > 0 \\ u(0) = a(q^m),\ \frac{du}{dt}(0) = b(q^m). \end{cases} \tag{7}$$

Under similar conditions on $a(q^m)$, $b(q^m)$ and $f(t;q^m)$ as in (41), the model state $u(t;q^m)$ exists uniquely, which satisfies the regularity (5). The regional identifiability problem using the model (7) is that: Under what conditions on q^m does the equality

$$u(t;q)|_{\Omega_0} = u(t;q^m)|_{\Omega_0}, \quad t > 0 \tag{8}$$

imply

$$q|_{\Omega_0} \ = \ q^m|_{\Omega_0} \quad ? \tag{9}$$

If the above implication holds true, the parameter q is said to be regionally identifiable in Ω_0 at q^m. By (6) the meaning of equality (8) (i.e., the zero output error on Ω_0) is that

$$u(t,x;q) = u(t,x;q^m) \quad for \ \ a.e.\ x \in \Omega_0,\ t > 0.$$

But the meaning of the regional identifiability (9) of q is somewhat ambiguous, because the space of parameters Q is not given exactly. The exact meaning of (9) depends on the choices of Q and Ω_0, and is explained in the practical problems of Section 3 and Section 4.

In this paper the case of regional identifiability including forcing functions is studied. Thus, for theoretical simplicity, it is assumed that $a(q) = a(q^m) = b(q) = b(q^m) = 0$, and

$$f(t;q) = f_0(t)f_1(q), \quad f_0 \in L^2_{loc}(0,\infty) \ \ f_1(q) \in H;$$

$$f(t;q^m) = f_0(t)f_1(q^m), \quad f_0 \in L^2_{loc}(0,\infty) \ \ f_1(q^m) \in H$$

in (4) and (7). The case including initial values (without forcing functions) can be treated by using cosine family as in this paper.

In order to study the regional identifiability problem in some general context, it is supposed that $A(q)$ is a Riesz *spectral* operator for each $q \in Q$ (cf. Curtain and Zwart [1]. The restriction of $A(q)$ on $H = L^2(\Omega)$ (H is independent of q!) defines a closed operator with dense domain

$$\mathcal{D}(A(q)) = \{\phi \in V(q):\ A(q)\phi \in H\}$$

in H, which is also denoted by the same simbol $A(q)$. The operator $A(q)$ is said to be a Riesz *spectral* operator on H if there exists a set of all isolated eigenvalues $\{\lambda_n(q)\}_{n=0}^{\infty}$ of $-A(q)$ such that the semigroup $T(t;q)$ generated by $-A(q)$ is represented by

$$T(t;q)\varphi = \sum_{n=0}^{\infty} e^{-\lambda_n(q)t} P_n(q)\varphi, \quad \varphi \in H. \tag{10}$$

Here in (10), $P_n(q)$ is the projection given by

$$P_n(q) = \frac{1}{2\pi i}\int_{\Gamma_n} (z - A(q))^{-1} dz \tag{11}$$

and Γ_n in (11) is a sufficiently small circle with center $-\lambda_n(q)$ such that its interior and Γ_n contain no points of the spectrum $\sigma(A(q))$ except for $-\lambda_n(q)$. Further we assume that $A(q)$ generates a strongly continuous cosine family $C(t;q)$ of bounded linear operators in H, $t \in (-\infty, \infty)$. (see Fattorini [5]). Then we can verify that the cosine family $C(t;q)$ and the sine family $S(t;q)$ are given by

$$C(t;q)\varphi = \sum_{n=0}^{\infty} \cos\sqrt{-\lambda_n(q)}t\ P_n(q)\varphi, \quad \varphi \in H \tag{12}$$

and

$$\begin{aligned} S(t;q)\varphi &= \int_0^t C(s;q)\varphi ds \\ &= \sum_{n=0}^{\infty} \frac{\sin\sqrt{-\lambda_n(q)}t}{\sqrt{-\lambda_n(q)}} P_n(q)\varphi, \quad \varphi \in H, \end{aligned} \tag{13}$$

respectively. The relation between the semigroup and the cosine family is given by

$$T(t;q)\varphi = \frac{1}{\sqrt{4\pi t}}\int_0^{\infty} \exp(-\frac{s^2}{4t})C(s;q)\varphi ds, \quad \varphi \in H, \quad t > 0 \tag{14}$$

Further the solution $u(t)$ of (4) is represented by

$$\begin{aligned} u(t;q) &= C(t;q)a(q) + S(t;q)b(q) \\ &\quad + \int_0^t S(t-s;q)f(s;q)ds, \quad t \geq 0. \end{aligned} \tag{15}$$

In this setting the semigroup $T(t;q)$ given by (10) is analytic in $t > 0$ (see [5]). Note that the series (10) at x

converge for a.e. $x \in \Omega$. It is well known that if $A(q)$ is nonnegative, selfadjoint on $L^2(\Omega)$ with compact resolvent, then $A(q)$ is a Riesz *spectral* operator which generates a strongly continuous cosine family in the sense given above.

In Nakagiri [9,10], Nakagiri and Yamamoto [11] and Yamamoto and Nakagiri [13,14] the identifiability problem is studied for the special case where the observation is total, i.e., $\Omega = \Omega_0$ in the framework of semigroup theory. But the operator theoretical method developed in the above references can not be applied to this regional identifiability problem, because the restriction operator to Ω_0 is not commutative with the operator $A(q)$.

The purpose of this paper is to establish a number of criteria for the regional identifiability of spatially-varying parameters in several types of hyperbolic distributed parameter systems. This paper is composed of five sections. Preliminary results are collected in Section 2. By using the results, the regional identifiability criteria are established in Section 3 and Section 4. In Section 3 the necessary and sufficient condition of parameters p_1, p_2, p_3, f_1 of the one dimensional general hyperbolic equations

$$u_{tt} = p_1(x)u_{xx} + p_2(x)u_x + p_3(x)u + f_0(t)f_1(x)$$

to be regionally identified in the subregion $\Omega_0 = [x_0, x_1] \subset \overline{\Omega} = [0,1]$ is established. In Section 4 the regional identifiability condition of p, f_1 for the multi dimensional wave equation

$$u_{tt} = \Delta u - p(x)u + f_0(t)f_1(x)$$

in $\Omega_0 \subset \overline{\Omega}$ is proved. Section 5 is the conclusions. We note that our approach can be applied to other types of hyperbolic equations such as beam equations, vibration equations of plates, Schrödinger equations.

2. Preliminary Results

In this section several preliminary results are stated without proofs. Some of the proofs can be found in [6], [11] and Levitan and Sargsjan [7]. The following result is taken from Titchmarsh [12; Theorem 152; p.325].

PROPOSITION 1 Let g and k belong to $L^1_{loc}(0,\infty)$. If k is not identically zero on $(0,\infty)$, and

$$g * k(t) = \int_0^t g(t-s)k(s)ds = 0 \quad \textit{for a.e. } t > 0, \tag{16}$$

then $g(t) = 0$ *for a.e.* $t \in (0,\infty)$.

The following proposition is crucial in our analysis.

PROPOSITION 2 Let $\{\lambda_n\}_{n=0}^\infty$ and $\{\mu_n\}_{n=0}^\infty$ be strictly monotonically increasing sequences such that $\sum_{n=0}^\infty c_n e^{-\lambda_n t}$ and $\sum_{n=0}^\infty d_n e^{-\mu_n t}$ converge uniformly on $[t_0,\infty)$ for each $t_0 > 0$. Assume that

$$\sum_{n=0}^\infty c_n e^{-\lambda_n t} = \sum_{n=0}^\infty d_n e^{-\mu_n t} \quad \text{for all } t > 0. \tag{17}$$

If $c_{n_0} \neq 0$ for some n_0, then there exists a k_0 such that $\lambda_{n_0} = \mu_{k_0}$ and $c_{n_0} = d_{k_0}$.

Let Q be the space of parameters defined on $\overline{\Omega} \subset \mathbf{R}^n$ and let $\Omega_0 \subset \overline{\Omega}$. From Proposition 2 the following useful proposition follows.

PROPOSITION 3 Assume that $A(q)$ is a Riesz *spectral* operator for each $q \in Q$ and that the series

$$T(t;q)\varphi(x) = \sum_{n=0}^\infty e^{-\lambda_n(q)t} P_n(q)\varphi(x) \quad \textit{a.e. } x \in \Omega_0 \tag{18}$$

satisfy the assumption in Proposition 2 for each $q \in Q$ and $\varphi \in L^2(\Omega)$. Assume further that $\varphi, \varphi^m \in L^2(\Omega)$ and

$$P_{n_0}(q^m)\varphi^m(x) \neq 0 \quad \textit{a.e. } x \in \Omega_1 \tag{19}$$

for some n_0 and $\Omega_1 \subset \Omega_0$. Then the equality

$$T(t;q)\varphi|_{\Omega_0} = T(t;q^m)\varphi^m|_{\Omega_0} \quad \text{for all } t > 0 \tag{20}$$

implies $\varphi|_{\Omega_0} = \varphi^m|_{\Omega_0}$ and that there exists a k_0 such that

$$\lambda_{n_0}(q^m) = \lambda_{k_0}(q), \tag{21}$$

$$P_{n_0}(q^m)\varphi^m(x) = P_{k_0}(q)\varphi(x) \quad \textit{a.e. } x \in \Omega_1. \tag{22}$$

In the applications given in Section 3 and Section 4 the conditions (21) and (22) imply the regional identifiability (9). By continuation, the equality (22) holds for all $x \in \overline{\Omega_1}$ provided that both functions in (22) are continuous on $\overline{\Omega}$.

Next we consider a bilinear form corresponding to one dimensional hyperbolic equation. Let $p_1(x) > 0$, $p_2(x)$, $p_3(x)$ and β_0, β_1 are functions and real numbers with $p_1 \in C^1[0,1]$, p_2, $p_3 \in C[0,1]$. Let $H = L^2(0,1)$ and $V = V(\beta_0,\beta_1)$ be a Hilbert space defined by

$$V = \Big\{\psi \in H^1(0,1) : -\frac{d\psi}{dx}(0) + \beta_0\psi(0) = \frac{d\psi}{dx}(1) + \beta_1\psi(1) = 0\Big\}.$$

Then (V,H) is a Gelfand triple space. The bilinear form $a(\phi,\psi)$ on $V \times V$ depending on p_1, p_2, p_3 is defined by

$$a(\phi,\psi) = (p_1\phi',\psi') - ((p_2 - p_1')\phi',\psi) - (p_3\phi,\psi). \tag{23}$$

It is verified that $a(\phi,\psi)$ satisfies the conditions (1) and (2). This form defines the operator $A \in \mathcal{L}(V,V')$ by (3) and the restriction on $H = L^2(0,1)$ is given by

$$\begin{cases} (-A\psi)(x) = p_1(x)\psi''(x) + p_2(x)\psi'(x) + p_3(x)\psi(x) \quad (0 < x < 1) \\ \mathcal{D}(A) = \Big\{\psi \in H^2(0,1) : -\frac{d\psi}{dx}(0) + \beta_0\psi(0) = \frac{d\psi}{dx}(1) + \beta_1\psi(1) = 0\Big\}. \end{cases} \tag{24}$$

A is not selfadjoint in $L^2(0,1)$ in general, however it is verified in the following Proposition 4 that A is a Riesz *spectral* operator and generates a strongly continuous cosine family on H. The eigenvalue problem for $-A$ is to find nonzero ψ and μ such that

$$\begin{cases} -A\psi(x) = \mu\psi(x) \qquad (0 < x < 1) \\ -\dfrac{d\psi}{dx}(0) + \beta_0\psi(1) = \dfrac{d\psi}{dx}(1) + \beta_1\psi(1) = 0 \end{cases} \tag{25}$$

holds.

PROPOSITION 4 There exists a system of eigenvalues and eigenfunctions $\{\mu_n, \psi_n\}_{n=0}^{\infty}$ of $-A$ in $L^2(0,1)$ such that $-A$ generates the semigroup $T(t)$ and the cosine family $C(t)$ on $L^2(0,1)$ given respectively by

$$T(t)\varphi = \sum_{n=0}^{\infty} e^{-\mu_n t} P_n\varphi, \quad \varphi \in L^2(0,1) \tag{26}$$

and

$$C(t)\varphi = \sum_{n=0}^{\infty} \cos\sqrt{-\mu_n} t\, P_n\varphi, \quad \varphi \in L^2(0,1). \tag{27}$$

Here

$$P_n\varphi = (\varphi, \psi_n)_\rho \psi_n = \left(\int_0^1 \varphi(x)\psi_n(x)\rho(x)dx\right)\psi_n \tag{28}$$

and ρ is the weight function defined by

$$\rho(x) = \frac{1}{p_1(x)} \exp\left(\int_0^x \frac{p_2(\xi)}{p_1(\xi)} d\xi\right) \quad (x \in [0,1]). \tag{29}$$

Further it is shown that the eigenfunctions ψ_n are uniformly bounded on $[0,1]$ and

$$\sqrt{\mu_n} = Cn + O\Big(\frac{1}{n}\Big) \text{ as } n \to \infty \tag{30}$$

for some nonzero constant C.

By Proposition 4 it is verified that the series $T(t)\varphi(x)$ at $x \in [0,1]$ in (26) converge uniformly on $[0,1] \times [t_0, \infty)$ for any $t_0 > 0$ by (30) and $\varphi \in L^2(0,1)$ (hence, a continuous function on $[0,1] \times (0,\infty)$).

PROPOSITION 5 Let ψ_{n_1}, ψ_{n_2} and ψ_{n_3} be three distinct eigenfunctions of $-A$. Let W be the Wronskian given by

$$W(\psi_{n_1}, \psi_{n_2}, \psi_{n_3})(x) = det \begin{pmatrix} \psi_{n_1}(x) & \psi_{n_2}(x) & \psi_{n_3}(x) \\ \psi'_{n_1}(x) & \psi'_{n_2}(x) & \psi'_{n_3}(x) \\ \psi''_{n_1}(x) & \psi''_{n_2}(x) & \psi''_{n_3}(x) \end{pmatrix}. \tag{31}$$

Then the set

$$E = \{x \in [0,1] : W(\psi_{n_1}, \psi_{n_2}, \psi_{n_3})(x) = 0\} \tag{32}$$

has no interior points in $[0,1]$.

The set E seems to be finite sets, however this could not be proved by the author. In the case where all ψ_n are analytic, this is clear. It is only shown that ψ_n are of C^∞-class by the Sobolev imbedding theorem.

3. One Dimensional Hyperbolic Equations

This section studies one dimensional general hyperbolic partial differential equations in $L^2(0,1)$. Throughout this section it is supposed that $\Omega = (0,1)$ and $\Omega_0 = [x_0, x_1] \subset \overline{\Omega} = [0,1]$, $x_0 < x_1$.

Consider the following system described by general hyperbolic partial differential equation of the form

$$\begin{cases} u_{tt} = p_1(x)u_{xx} + p_2(x)u_x + p_3(x)u + f_0(t)f_1(x) \\ \qquad\qquad\qquad\qquad (x \in \Omega,\ t > 0) \\ u(x,0) = 0, \quad u_t(x,0) = 0 \qquad\qquad (x \in \Omega) \\ -u_x(0,t) + \beta_0 u(0,t) = u_x(1,t) + \beta_1 u(1,t) = 0 \\ \qquad\qquad\qquad\qquad (t > 0), \end{cases} \tag{33}$$

where $p_1(x) > 0$, $p_2(x)$, $p_3(x)$, β_0, β_1 and $f_1(x)$ are unknown parameters with $p_1 \in C^1[0,1]$, $p_2 \in C[0,1]$, $p_3 \in C[0,1]$ to be regionally identified on $\Omega_0 = [x_0, x_1] \subset \Omega$. Let $C^1_+[0,1] = \{\varphi \in C^1[0,1] : \varphi(x) > 0 \text{ for all } x \in [0,1]\}$. For the system (33) the parameter space Q is defined by

$$Q = C^1_+[0,1] \times C[0,1] \times C[0,1] \times L^2(0,1) \times \mathbf{R} \times \mathbf{R} \tag{34}$$

and the unknown parameter q is given by

$$q = (p_1, p_2, p_3, f_1, \beta_0, \beta_1) \in Q. \tag{35}$$

The model parameter q^m is given by

$$q^m = (p_1^m, p_2^m, p_3^m, f_1^m, \beta_0^m, \beta_1^m) \in Q \tag{36}$$

and the model system is defined by (33) in which q is replaced by q^m.

The restriction $q|_{\Omega_0}$ of q on $\Omega_0 = [x_0, x_1]$ is defined as follows:

If $x_0 \neq 0, x_1 \neq 1$, then

$$q|_{\Omega_0} = (p_1|_{\Omega_0}, p_2|_{\Omega_0}, p_3|_{\Omega_0}, f_1|_{\Omega_0});$$

if $x_0 = 0, x_1 \neq 1$, then

$$q|_{\Omega_0} = (p_1|_{\Omega_0}, p_2|_{\Omega_0}, p_3|_{\Omega_0}, f_1|_{\Omega_0}, \beta_0);$$

if $x_0 \neq 0, x_1 = 1$, then

$$q|_{\Omega_0} = (p_1|_{\Omega_0}, p_2|_{\Omega_0}, p_3|_{\Omega_0}, f_1|_{\Omega_0}, \beta_1);$$

and if $x_0 = 0, x_1 = 1$ (i.e., $\Omega_0 = \Omega$), then

$$q|_{\Omega_0} = q = (p_1, p_2, p_3, f_1, \beta_0, \beta_1).$$

Hence, the regional identifiability (9) means that

$$p_i(x) = p_i^m(x) \quad (x \in \Omega_0, \quad i = 1,2,3)$$

$$f_1(x) = f_1^m(x) \quad (a.e.\ x \in \Omega_0)$$

and

$$\beta_0 = \beta_1^m$$

when $\Omega_0 = [0, x_1]$, $x_1 \neq 1$. Other cases are similar.

The system (33) and the model system can be described by the second order evolution equation systems (4) and (7) in some Gelfand triple spaces with $f(t;q) = f_0(t)f_1$, $f(t;q^m) = f_0(t)f_1^m$ and $a(q) = a(q^m) = b(q) = b(q^m) = 0$, respectively. Then the operator $A = A(q)$ defined through the bilinear form (23) on $V(q) = V(\beta_0, \beta_1)$ correspondig to the system (33) is characterized by (24) and has compact resolvent in $L^2(0,1)$. Similar facts holds for the model operator $A^m = A(q^m)$ corresponding to the model system. Then by Propsition 4 there are the sets of real eigenvalues and eigenfunctions $\{\lambda_n, \varphi_n\}_{n=0}^{\infty}$ of $-A^m$ and $\{\mu_n, \psi_n\}_{n=0}^{\infty}$ of $-A$ such that A^m and A are Riesz *spectral* operators and generates strongly continuous cosine families. The weight functions ρ and ρ^m are defined by (29) and

$$\rho^m(x) = \frac{1}{p_1^m(x)} \exp\left(\int_0^x \frac{p_2^m(\xi)}{p_1^m(\xi)} d\xi\right) \quad (x \in [0,1]), \quad (37)$$

respectively. The semigroup $T(t)$ generated by $-A$ is given by (26) with (28), and the cosine family $C(t)$ is given by (27) with (28). Also $-A^m$ generates the semigroup $T^m(t)$ and the cosine family $C^m(t)$ given respectively by

$$T^m(t)\varphi = \sum_{n=0}^{\infty} e^{-\lambda_n t}(\varphi, \varphi_n)_{\rho^m}\varphi_n, \quad \varphi \in H \quad (38)$$

and

$$C^m(t)\varphi = \sum_{n=0}^{\infty} \cos\sqrt{-\lambda_n}t\,(\varphi, \varphi_n)_{\rho^m}\varphi_n, \quad \varphi \in H. \quad (39)$$

The corresponding sine families $S^m(t)$ and $S(t)$ are give by

$$S^m(t)\varphi = \sum_{n=0}^{\infty} \frac{\sin\sqrt{-\lambda_n}t}{\sqrt{-\lambda_n}}(\varphi, \varphi_n)_{\rho^m}\varphi_n, \quad \varphi \in H, \quad (40)$$

$$S(t)\varphi = \sum_{n=0}^{\infty} \frac{\sin\sqrt{-\mu_n}t}{\sqrt{-\mu_n}}(\varphi, \psi_n)_{\rho}\psi_n, \quad \varphi \in H, \quad (41)$$

respectively. Hence, in this case the equality (8) is equivalent to

$$\int_0^t F(x, t-s) f_0(s) ds = 0 \quad (a.e.\ x \in \Omega_0,\ t > 0), \quad (42)$$

where

$$F(x,t) = \sum_{n=0}^{\infty} \frac{\sin\sqrt{-\lambda_n}t}{\sqrt{-\lambda_n}}(f_1^m, \varphi_n)_{\rho^m}\varphi_n(x) - \sum_{n=0}^{\infty} \frac{\sin\sqrt{-\mu_n}t}{\sqrt{-\mu_n}}(f_1, \psi_n)_{\rho}\psi_n(x). \quad (43)$$

If $f_0(t)$ is not identically 0 in $(0,\infty)$, then by Proposition 1 it follows from (42) that $F(x,t) = 0$ for $a.e.\ x \in \Omega_0 = [x_0, x_1]$ and all $t > 0$. By differentiating this equality we have from (27) and (39) and

$$C(t)f_1|_{\Omega_0} = C^m(t)f_1^m|_{\Omega_0} \quad \text{for all } t > 0. \quad (44)$$

Applying the transformation (14) to (44) and noting that the restriction operator χ_{Ω_0} is bounded, we can verify

$$T(t)f_1|_{\Omega_0} = T^m(t)f_1^m|_{\Omega_0} \quad \text{for all } t > 0. \quad (45)$$

Hence by Proposition 3 and Proposition 5 we can prove the following Theorem 1.

THEOREM 1 Assume that f_0 is not identically 0 in $(0,\infty)$. Then the parameter q in (34) is regionally identifiable in Ω_0 at q^m if and only if there exist three distinct eigenfunctions $\varphi_{n_1}, \varphi_{n_2}, \varphi_{n_3}$ of $-A^m$ such that

$$(f_1^m, \varphi_{n_i})_{\rho^m} \neq 0 \quad (i = 1,2,3). \quad (46)$$

That is, the condition (46) is necessary and sufficient for that the equality (6) implies

$$p_i(x) = p_i^m(x) \quad (x \in \Omega_0,\ i = 1,2,3) \quad (47)$$

and

$$f_1(x) = f_1^m(x) \quad (\text{a.e. } x \in \Omega_0) \quad (48)$$

if $x_0 \neq 0$, $x_1 \neq 1$, and further implies $\beta_0 = \beta_0^m$ (resp. $\beta_1 = \beta_1^m$) if $x_0 = 0$ (resp. $x_1 = 1$).

4. Multi Dimensional Wave Equations

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. The observable subregion $\Omega_0 \subset \overline{\Omega}$ is assumed to be closed and connected having positive Lebesgue measure in $\mathbf{R}^n$. Further it is assumed that the intersection $\Omega_0 \cap \Gamma$ is also closed and connected in Γ. In this section the regional identifiability of systems described by the following multi dimensional heat equations is studied.

Consider the system described by the n-dimensional wave equation with 0 initial value and Robin-type boundary condition

$$\begin{cases} u_{tt} = \Delta u - p(x)u + f_0(t)f_1(x) & (x \in \Omega,\ t > 0) \\ u(x,0) = u_t(x,0) = 0 & (x \in \Omega) \\ \frac{\partial u}{\partial n} + \beta u\big|_{\partial\Omega} = 0 & (t > 0) \end{cases} \quad (49)$$

and the model system described by

$$\begin{cases} u_{tt} = \Delta u - p^m(x)u + f_0(t)f_1^m(x) & (x \in \Omega,\ t > 0) \\ u(x,0) = u_t(x,0) = 0 & (x \in \Omega) \\ \frac{\partial u}{\partial n} + \beta^m u\big|_{\partial\Omega} = 0 & (t > 0), \end{cases} \quad (50)$$

where p, $p^m \in C(\overline{\Omega})$, β, $\beta^m \in C(\Gamma)$ and $\frac{\partial}{\partial n}$ denotes the differentiation along the outer unit conormal vectors,

$f_0 \in L^1(0,T)$, and f_1, $f_1^m \in L^2(\Omega)$. Here in the system (49) a potential $p(x)$ in $C(\overline{\Omega})$, a boundary coefficient $\beta \in C(\Gamma)$ and the forcing function $f_1(x)$ in $L^2(\Omega)$ are unknown quantities to be regionally identified in Ω_0 by using the model system (50). The parameter space Q is defined by

$$Q = C(\overline{\Omega}) \times L^2(\Omega) \times C(\Gamma), \tag{51}$$

and the unknown and the model parameters q and q^m are given respectively by

$$q = (p, f_1, \beta), \quad q^m = (p^m, f_1^m, \beta^m) \in Q \tag{52}$$

in this case. The restriction $q|_{\Omega_0}$ on Ω_0 is defined by

$$\begin{cases} q|_{\Omega_0} = (p|_{\Omega_0}, f_1|_{\Omega_0}) & \text{if} \quad meas(\Omega_0 \cap \Gamma) = 0 \\ q|_{\Omega_0} = (p|_{\Omega_0}, f_1|_{\Omega_0}, \beta|_{\Omega_0 \cap \Gamma}) & \text{if} \quad meas(\Omega_0 \cap \Gamma) > 0, \end{cases} \tag{53}$$

where $meas(\Omega_0 \cap \Gamma)$ is the Lebesgue measure on Γ. The systems (49) and (50) are described by (4) and (7) in some Gelfand triple spaces.

By using a priori estimates of elliptic equations, unique continuation theorem due to Carderón we can prove the following theorem.

THEOREM 2 Assume that f_0 is not identically 0 in $(0, \infty)$. Then the parameter $q \in Q$ in (51) is regionally identifiable in Ω_0 at q^m if and only if

$$f_1^m \text{ is not identically zero in } L^2(\Omega). \tag{54}$$

That is, the condition (54) is necessary and sufficient for that the equality (6) implies

$$p(x) = p^m(x) \quad (x \in \Omega_0) \tag{55}$$

and

$$f_1(x) = f_1^m(x) \quad (\text{a.e. } x \in \Omega_0) \tag{56}$$

if $meas(\Omega_0 \cap \Gamma) = 0$, and further implies

$$\beta(y) = \beta^m(y) \quad (y \in \Omega_0 \cap \Gamma) \tag{57}$$

if $meas(\Omega_0 \cap \Gamma) > 0$.

5. Conclusions

The regional identifiability of spatially-varying parameters in distributed systems of hyperbolic types is formulated by the second order evolution equations setting via semigroup and cosine family methods. Using the setting, necessary and sufficient conditions for the regional identifiability of one dimensional general hyperbolic equations and multi-dimensional wave equations are established. Such conditions are rather simple, and therefore are easily applied to practical systems studied in this paper. The method presented here can be applied to various types of hyperbolic distributed parameter systems including beam equations and vibrating plate equations.

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