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Robust Controller Design for Parametrically Uncertain System

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Abstract

The design problem of the control system is the ability to synthesize controller that achieve robust stability and robust performance. The paper explains the Finite Inclusions Theorem (FIT) by the procedure namely FIT synthesis. It is developed for synthesizing robustly stabilizing controller for parametrically uncertain system. The fundamental problem in the study of parametrically uncertain system is to determine whether or not all the polynomials in a given family of characteristic polynomials is Hurwitz i.e.,all their roots lie in the open left-half plane.

By FIT it can prove a polynomial is Hurwitz from only approximate knowledge of the polynomial's phase at finitely many points along the imaginary axis.

An example shows the simplicity of using the FIT synthesis to directly search for robust controller of parametrically uncertain system by way of solving a sequence of systems of linear inequalities. The systems of inequalities are solved via the projection method which is an elegantly simple technique for solving (finite or infinite) systems of convex inequalities in an arbitrary Hilbert space. Results from example show that the controller synthesized by FIT synthesis is better than by H_∞ synthesis with parametrically uncertain system as well as satisfied the objectives for a considerably larger range of uncertainty.

1. Introduction

Modeling uncertainties affect the system in many different ways: degradation of the system performance and destabilization are the two most important adverse effects.

In this paper we deal with the linear time-invariant single-input single-output problem of controller synthesis for guaranteed robust stability and performance when the plant includes parameter uncertainty [5]. One method of incorporating uncertainty in system models is by using a parametric description [1]. Recent development in the analysis of systems with parametric uncertainty have been implied by Kharitonov's theorem [5].

Let $\phi(s,a)$ be the closed loop characteristic polynomial with coefficients that depend on the vector parameter **a**, which lies in a some set: $\Omega_a = \{a \in R^k | a_i^- \le a_i \le a_i^+, 1 \le i \le k\}, a_i^- < 0, a_i^+ > 0, 1 \le i \le k$, of a unity feedback configuration. A recent result, the Finite Inclusions Theorem (FIT) [2,3,4], says that robust stability of a polynomial family can be determined from the location of a finite number of the value set $\phi(j\omega, \Omega_a)$ at appropriately chosen frequencies.

We will first state the FIT and present an iterative synthesis algorithm based on FIT. Controller parameters are improved at each iteration are computed by solving linear inequalities [3,4,8]. The procedure based on FIT seems to have many advantages over other methods. First of all there is a trade off between conservativeness of the solution and the number of the inequalities to be solved by a computer and this can be controlled by the designer. The controller order way be fixed ahead of time, so we can avoid high order controllers usually generated by H_∞. An example will be used to demonstrate the procedure.

2. Formulation

Theorem 1 *(finite inclusions theorem) Let* $p(s,a) = \sum_{j=0}^{0} p_i(a)s^j$, $a \in \Omega_a$ $n \ge 0$, *and* $p_i : \Omega_a \to$ **C**. *Further, Let* $\Gamma \subset$ **C** *be a closed Jordan curve such that int* Γ *is convex. Then for all* $a \in \Omega_a$, $p(s,a)$ *is of degree n and has all its roots in int* Γ *if there exists* $m \ge 1$ *intervals* $(c_k, d_k) \subset$ **R** *and a real* ω_k, $1 \le k \le m$ *such that*

$$0 \le \omega_1 \le \omega_2 \le \omega_3 \le \ldots \le \omega_m$$

$$\frac{-\pi n}{2} - \pi \le c_1 < d_1 \le -\frac{\pi n}{2} + \pi$$

$$\frac{\pi n}{2} - \pi \le c_m < d_m \le \frac{\pi n}{2} + \pi$$

$$\forall \quad 1 \le k \le m \quad \max\{d_k - c_{k+1}, d_{k+1} - c_k\} \le \pi$$

$$\forall \quad 1 \le k \le m \quad p(j\omega_k, \Omega_a) \in \{re^{j\theta} \mid r > 0, c_k < \theta < d_k\}$$

Let $C(s) = \dfrac{n_c(s)}{d_c(s)}$ is a proper controller. The sensitivity transfer function is given by $S(s,a) = (1 + P(s,a)C(s))^{-1}$ where $\phi(s,a) = d_c(s)d_p(s,a) + n_c(s)n_p(s,a)$ is the closed loop characteristic polynomial. For robust asymptotic tracking one has design specification: Find a controller $C(s)$ that robustly stabilizes the loop and guarantees that $|S(j\omega,a)| < |W_1(j\omega)|^{-1}$, $\forall\ \omega \in [0,\infty)$ and $a \in \Omega_a$. $W_1(s) = \dfrac{n_w(s)}{d_w(s)}$ is a given strictly proper, stable, minimum phase, transfer function which weights the sensitivity transfer function. This requirement can be equivalently stated as

$$|W_1(j\omega)\, S(j\omega,a)| < 1 \quad \forall \ \omega \in [0,\infty), \quad a \in \Omega_a$$

After several steps [2] one can show that a necessary and sufficient condition for robust asymptotic tracking is the stability of the polynomial family

$$\psi(s,a,r,\alpha) = re^{j\alpha} n_w(s) d_c(s) d_p(s,a) + d_w(s)\phi(s,a)$$

for all $\alpha \in [0,2\pi), r \in [0,1]$ and $a \in \Omega_a$. Let $\Omega_q = \{q = (a,r,\alpha) \mid a \in \Omega_a, r \in [0,1], \alpha \in [0,2\pi)\}$. Stability of $\psi(s,a,r,\alpha)$ is necessary as well. Now the polynomial $\psi(s,a,r,\alpha)$ is in the form required by the Finite Inclusions Theorem. In view of Theorem 1 and the above discussion we have

Theorem 2 *(Robust Performance) A degree q proper controller C(s) with monic denominator robustly stabilizes the family of degree $\bar{n}$ plants $P(s,a) = \dfrac{n_p(s,a)}{d_p(s,a)}$ and makes $|W_1(j\omega)\, S(j\omega,a)| < 1$, $\forall\, \omega \in [0,\infty)$ and $a \in \Omega_a$ if for some $m \geq 1$ there exist real $\omega_k, c_k, d_k, c_k < d_k, 1 \leq k \leq m$ such that*

$$0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq \ldots \leq \omega_m$$

$$\frac{-\pi n}{2} - \pi \leq c_1 < d_1 \leq -\frac{\pi n}{2} + \pi$$

$$\frac{\pi n}{2} - \pi \leq c_m < d_m \leq \frac{\pi n}{2} + \pi$$

$$\forall \quad 1 \leq k \leq m \quad \max\{d_k - c_{k+1}, d_{k+1} - c_k\} \leq \pi$$

$$\forall \quad 1 \leq k \leq m \quad \psi(j\omega_k, \Omega_q) \in \{re^{j\theta} \mid r > 0, c_k < \theta < d_k\}$$

3. An Algorithm For Robust Performance Synthesis

We are now ready to suggest a robust performance synthesis algorithm based on Theorem 2 [2]. Let the order q controller be parameterized as

$$C(s) = \frac{x_{2q+1}s^q + x_{2q}s^{q-1} + \ldots + x_{q+1}}{s^q + x_q s^{q-1} + x_{q-1}s^{q-2} + \ldots + x_1} = \frac{n_c(s)}{d_c(s)}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_{2q+1}) \in \mathbf{R}^d$ and $d = 2q+1$. Robust performance will be achieved if the polynomial family

$$\psi(s,a,b) = b n_w(s) d_c(s) n_p(s,a) + d_w(s)\phi(s,a)$$

are simultaneously stabilized for $a \in \Omega_a$ and **b** in the unit disc. Since **b** takes values in the unit disc the value sets for $\phi(s,a)$ will not be polygonal. A FIT based algorithm can be computationally enhanced if these value sets are polygons at each frequency. This can be easily achieved if some polygonal overbound is employed for the unit disc. This can be done arbitrarily closely by using higher order polygonal overbounds. Here we will use the simplest one (see Figure 1), the unit square Ω_{bs} where the complex number $\mathbf{b} = b_1 + jb_2$

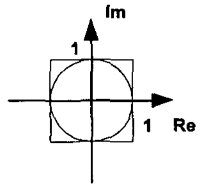

Fig. 1. Square Overbound of a Disc

Algorithm

STEP 1 Let $\mathbf{x}^{(1)} \in \mathbf{R}^d$ and $\Omega_{ab}^{(1)} \subset \Omega_{ab}$ be such that $\phi(s, \Omega_{ab}^{(1)}, x^{(1)})$ is stable and set j:= 1.

STEP 2 Determine $m^{(j)} \geq 1$ sectors $S_k^{(j)}, 1 \leq k \leq m^{(j)}$, and frequencies $\omega_k^{(j)}$ along the $j\omega$ axis such that $\phi(\omega_k^{(j)}, Ext\Omega_{ab}^{(j)}, x^{(j)}) \subset S_k^{(j)}$. By FIT, $\phi(s, \Omega_{ab}^{(j)}, x^{(j)})$ is stable. Each $\omega_k^{(j)}$ should roughly center (angularly) the set $\phi(\omega_k^{(j)}, Ext\Omega_{ab}^{(j)}, x^{(j)})$ in $S_k^{(j)}$.

STEP 3 Choose a slightly larger set $\Omega_{ab}^{(j+1)} \supset \Omega_a^{(j)}$. First this should affect the **b** parameters. When (if) $\Omega_b^{(j)} \supset \Omega_{bs}$, the enlargement in the **b**-direction terminates and the enlargement in the **a**-direction commences.

STEP 4 Compute a new vector of controller parameters $x^{(j+1)}$ such that $\phi(j\omega_k^{(j)}, Ext\Omega_{ab}^{(j+1)}, x^{(j+1)}) \subset S_k^{(j)}$ for all k. Note, this is equivalent to solving a system of linear inequalities in $x^{(j+1)}$. If no solutions exist to this system of inequalities, return to step 3 and choose a smaller $\Omega_{ab}^{(j+1)}$.

STEP 5 Let $j := j+1$, and if $\Omega_{ab}^{(j)} \supset \Omega_{ab}$, stop; otherwise, go to Step 2.

4. Example

The robust performance objectives for the system are as follows

- P.O. ≤ 20%, the overshoot must be less than 20% (coresponding to a damping ratio greater than 0.5)
- $T_s \leq 2$ seconds, the step response must reach a ±2% envelope by 2 seconds.

Consider the following system with nominal plant

$$P_0(s) = \frac{(s+1)(s+10)}{(s+2)(s+4)(s-1)}$$

and multiplicative uncertainty

$$\Delta_m(s,a) = \frac{a_1(5.5s+7) + a_2(s^2+0.5s+2)}{(s+1)(s+10)}$$

where $P(s,a) = P_0(s)(1+\Delta_m(s,a))$, $\mathbf{a} = (a_1, a_2)$ lies is the rectangle $\Omega_a = \{(a_1,a_2) \mid -\alpha < a_1 < \alpha, -\alpha < a_2 < \alpha\}$. The positive parameter α regulates the size of the rectangle.

The performance objectives are robust stability and robust asymptotic tracking where the bound on the sensitivity transfer function is generated by

$$W_1^{-1}(s) \quad = \quad 0.5(20s+1)^2$$

We would like to achieve these objectives for as large a value of α as possible. As mentioned in the introduction we would like to compare FIT synthesis with other methodologies and in this paper we chose H_∞. The following steps show the procedure of H_∞ synthesis.

Consider the transfer function of a family plant with multiplicative uncertainty

$$P(s) \quad = \quad P_0(s)(1+\Delta W_2(s))$$

where $W_2(s)$ is some given minimum phase stable transfer function which weights the complementary sensitivity transfer function $T(s)$ and Δ is any arbitrary stable and proper transfer function with $|\Delta(j\omega)| \le 1$, for all $\omega \in [0,\infty)$. A necessary and sufficient condition for robust stability is $|W_2(j\omega)T(j\omega)| < 1$, for all $\omega \in [0,\infty)$. This elegant characterization of robust stability allows for controller synthesis as a solution to a Nevanlina-Pick method [6,7].

A bound $W_2(j\omega)$ for the magnitude of $\Delta_m(j\omega, a_1, a_2)$ was generated

$$W_2(s) \quad = \quad \frac{1.13366082837215s^2 + 6.88573032033468s + 9}{(s+1)(s+10)}$$

A necessary and sufficient condition for robust stability and performance for the loop with norm bounded uncertainty (assuming nominal stability) is

$$\left\| |W_1(j\omega)S(j\omega)| + |W_2(j\omega)T(j\omega)| \right\| < 1 \quad \forall\ \omega \in [0,\infty)$$

In our case we want to find the largest α for which robust stability and robust performance can be achieved. This was accomplished by multiplying $W_2(s)$ by α, solving the corresponding H_∞ problem. The controller for the largest α was found to be

$$n_{c\infty}(s) = 8450443.55s^6 + 144384183s^5 + 722201690s^4 + 13111899190s^3 + 784544834s^2 + 59833785.9s + 920190.862$$

$$d_{c\infty}(s) = s^7 + 952191.883s^6 + 20260790.4s^5 + 111695492s^4 + 3.6955730.5s^3 - 6.1657555.7s^2 - 6.387707.17s - 161313.783$$

for the parameter set

$\Omega_a = \{a \in R^2 | a_1 \in [-0.8399, 0.8399],\ a_2 \in [-0.8399, 0.8399]\}$.

Figure 2 and 3 show the plot of the magnitude $W_1(j\omega)$ against $S(j\omega)$ and the magnitude of $W_2(j\omega)$ against $T(j\omega)$ respectively.

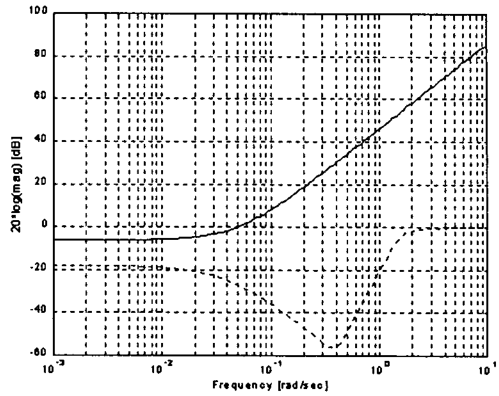

Fig. 2. the plot of the magnitude of $W_1(j\omega)$ against $S(j\omega)$

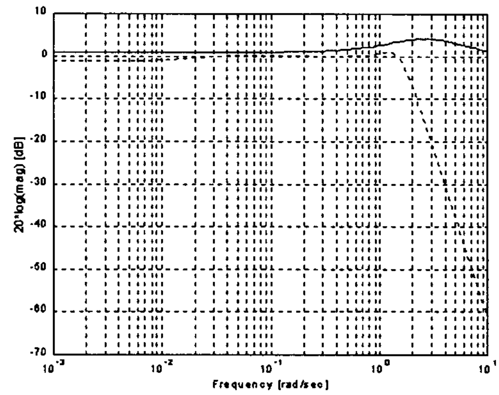

Fig. 3. the plot of the magnitude of $W_2(j\omega)$ against $T(j\omega)$

The following steps show the procedure of FIT synthesis. For this example the polynomial $\psi(s,a,b)$ that needs to be robustly stabilized is given by

$$\psi(s,a,b) \quad = \quad d_c(s)d_p(s,a,b) + n_c(s)n_p(s,a,b)$$

where

$$n_p(s,a,b) = (400a_2 + 400)s^4 + (2200a_1 + 240a_2 + 4440)s^3 + (3020a_1 + 821a_2 + 4441)s^2 + (285.5a_1 + 80.5a_2 + 411)s + 7a_1 + 2a_2 + 10$$

$$d_p(s,a,b) \quad = \quad 400s^5 + 2040s^4 + (2b+1001)s^3 + (10b-3115)s^2 + (4b-318)s - 16b - 8$$

The Algorithm requires that an initial stabilizing controller be entered. This was

$$C(s) \quad = \quad \frac{3.6589061s^2 + 21.9534363s + 29.2712485}{(s-5.3389732)(s+100)}$$

The resulting controller is

$$n_{cFIT}(s) = 366.1450765s^2 + 2.1968705e+3s + 1.0803123e+4$$

$$d_{cFIT}(s) = s^2 + 94.6610268\,s + 3.8269760\,e-4$$

which guarantees robust performance for not only the given range but for the parameter set $\Omega_a = \{a \in R^2 | a_1 \in [-1.1102, 1.1102], a_2 \in [-1.1102, 1.1102]\}$. The two plot that follow show the results of the design. Figure 4 shows the step response of the family of plant for the time period up to 2 seconds. It can be seen that the overshoot is less than 20% and that response is inside the ±2% envelope less than 2 seconds. Figure 5 shows response of output to disturbance.

5. Conclusion

The synthesized robust controller is based on the Finite Inclusions Theorem has been proposed in this paper. In example this method gave good results when compared with H_∞ synthesis. The controller can also be synthesized to satisfy the desired specifications. Fast rejection of disturbance is also obtained.

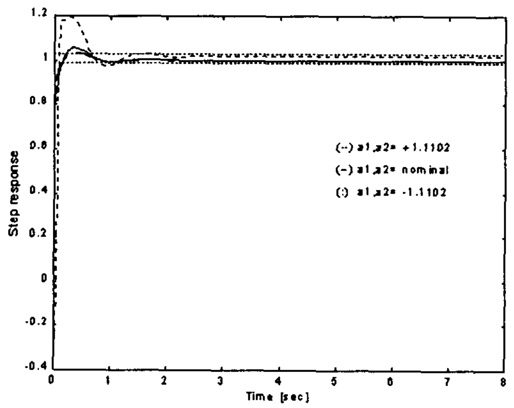

Fig. 4. Step response

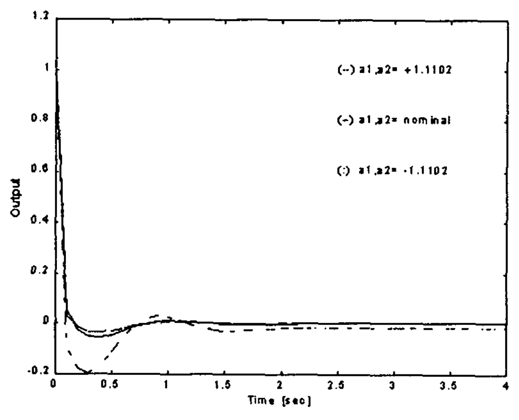

Fig. 5. Response of output to disturbance

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