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Sliding Mode Control with Finite Time Error Convergence

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Abstract

In this paper, a sliding mode controller guaranteeing finite time error convergence is proposed for uncertain systems. By using a novel sliding hyperplane, it is guaranteed that the output tracking error converges to zero in finite time.

Keywords: Sliding Mode Control, Uncertain Systems, Robust Control.

1 Introduction

A normal approach of control system designer is to make a model of the plant under consideration and then design a controller for the model so that the closed-loop system is stable. Actually, almost all of cases, there exists a modelling error. In addition, it is very difficult to measure the actual parameter of the plant *exactly*. It must has the measurement error/uncertainty. Thus, the control system that can handle the system uncertainty has been required.

It has been known that the sliding mode control, which is also known as variable structure control, has robust and invariant property to parameter uncertainties and external disturbances [1]-[3]. The sliding mode control is designed for the system state to be forced to stay on the predetermined sliding surface. When the system is in the sliding mode, the overall system shows the invariance property to parameter variations and external disturbances, and the dynamics of the closed-loop system is determined by the prescribed sliding surface.

Generally, a sliding surface is designed as a linear dynamic equation, *e.g.*, $s = \dot{e} + c\,e$. However, the linear sliding surface can guarantee only the *asymptotic* error convergence in the sliding mode, *i.e.*, the output error can not converge to zero in *finite* time.

Recently, to get a better performance on the sliding mode, terminal sliding mode control methods have been studied [4]-[7]. They have used nonlinear functions, $s = \dot{e} + c \cdot e^r$, where $c > 0$ is a positive constant, and $0 < r < 1$ is a rational number, so that the error converges to zero in *finite* time. However, these methods have a singularity problem [8]. It is a critical one because singular points are located around the origin(the equilibrium point) in the state space, that is, a set of singular points is $\{ (e, \dot{e}) \mid e = 0 \text{ and } \dot{e} \neq 0 \}$. Therefore, conventional terminal sliding mode control schemes may cause a problem – very large control signal – in the steady state, or when an initial condition is located around the set of singular points – a vertical axis in the phase plane. Although a modified terminal sliding mode control method was proposed [8], it also has the same problem because it does not always generate a bounded range space for any bounded domain.

Thus, a novel sliding mode control method is proposed in this paper. The proposed sliding mode control scheme uses two stage control; sliding mode control with a conventional linear sliding hyperplane, and that with a terminal sliding hyperplane. By switching the sliding hyperplane appropriately, it is guaranteed that the tracking error converges to zero in *finite* time with *no singularity*.

2 Problem Formulation

Consider a second-order nonlinear uncertain system described by

$$\ddot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{G}(\mathbf{x}, \dot{\mathbf{x}}, t)\mathbf{u}(t), \qquad (1)$$

where $\mathbf{x} \in \Re^n$ is the state vector, $\mathbf{u} \in \Re^n$ is the control input vector, $\mathbf{f} \in \Re^n$ is the vector of nonlinear dynamics composed of f_i, and the matrix $\mathbf{G} \in \Re^{n\times n}$ is the control gain, and is assumed to be invertible and matching condition is satisfied. It is assumed that the following equations are satisfied:

$$\left|\hat{f}_i - f_i\right| \leq F_i \qquad \text{and} \qquad \mathbf{G} = (\mathbf{I} + \mathbf{\Delta})\hat{\mathbf{G}},$$

where $i = 1, 2, \cdots, n$, $\mathbf{I}$ is the $n \times n$ identity matrix, and the estimated input matrix $\hat{\mathbf{G}}$ is assumed to be invertible, and $\mathbf{\Delta} \in \Re^{m\times m}$ composed of Δ_{ij} satisfies the following inequality:

$$|\Delta_{ij}| \leq \overline{\Delta}_{ij},$$

where $\overline{\Delta}_{ij} > 0$, and $\|\overline{\mathbf{\Delta}}\| < 1$, $\hat{(\cdot)}$ represents a nominal/estimated value of $(\cdot)$, $i = 1, 2, \cdots, n$, and $j = 1, 2, \cdots, n$.

3 Control System Design

3.1 The First Stage

In the first stage, a conventional sliding mode control system is designed for the given plant. Let us define a linear sliding hyperplane $s(t)$ as

$$\mathbf{s}^L(t) = \dot{\mathbf{e}}(t) + \Lambda \mathbf{e}(t), \quad (2)$$

where $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x_d}(t)$, $\mathbf{x_d}(t)$ is a given twice continuously differentiable reference trajectory, $\Lambda = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ with $\lambda_i > 0$, where $i = 1, 2, \cdots, n$.

By using the linear sliding surface (2), the following theorem can be derived for the existence of the sliding mode.

Theorem 1 *The existence of the sliding mode can be guaranteed if the following controller is applied to the plant (1):*

$$\mathbf{u}^L(t) = \hat{\mathbf{G}}^{-1}\left[\ddot{\mathbf{x}}_\mathbf{d} - \Lambda\dot{\mathbf{e}} - \hat{\mathbf{f}} - \mathbf{k} \bullet \mathbf{sgn}(\mathbf{s})\right], \quad (3)$$

where $\mathbf{k}\bullet\mathbf{sgn}(\mathbf{s})$ *is the vector of components* $k_i{\cdot}sgn(s_i)$*, i. e.,*

$$\mathbf{k}\bullet\mathbf{sgn}(\mathbf{s}) = [k_1{\cdot}sgn(s_1),\ k_2{\cdot}sgn(s_2),\ \cdots,\ k_n{\cdot}sgn(s_n)],$$

and each positive constant k_i *is chosen such that the following equation is satisfied.*

$$\begin{aligned}(1 - \overline{\Delta}_{ii})\, k_i - \sum_{j\neq i}^{n} \overline{\Delta}_{ij} k_j \\ = F_i + \sum_{j=1}^{n} \overline{\Delta}_{ij} \left|\ddot{x}_{dj} - \lambda_j \dot{e}_j - \hat{f}_j\right| + \eta_i, \quad (4)\end{aligned}$$

where $\eta_i > 0$*, and* $i = 1, 2, \cdots, n$.

Proof Differentiating the linear sliding surface $\mathbf{s}^L$ in (2) with respect to time, one can obtain

$$\begin{aligned}\dot{\mathbf{s}}^L &= \ddot{\mathbf{x}} - \ddot{\mathbf{x}}_\mathbf{d} + \Lambda\dot{\mathbf{e}} = \mathbf{f} + \mathbf{G}\mathbf{u}^L - \ddot{\mathbf{x}}_\mathbf{d} + \Lambda\dot{\mathbf{e}} \\ &= \mathbf{f} - \ddot{\mathbf{x}}_\mathbf{d} + \Lambda\dot{\mathbf{e}} \\ &\quad + (\mathbf{I} + \mathbf{\Delta})\,\hat{\mathbf{G}}\hat{\mathbf{G}}^{-1}\left[\ddot{\mathbf{x}}_\mathbf{d} - \Lambda\dot{\mathbf{e}} - \hat{\mathbf{f}} - \mathbf{k}\bullet\mathbf{sgn}(\mathbf{s})\right] \\ &= \mathbf{f} - \hat{\mathbf{f}} - \mathbf{k}\bullet\mathbf{sgn}(\mathbf{s}) \\ &\quad + \mathbf{\Delta}\left(\ddot{\mathbf{x}}_\mathbf{d} - \Lambda\dot{\mathbf{e}} - \hat{\mathbf{f}} - \mathbf{k}\bullet\mathbf{sgn}(\mathbf{s})\right).\end{aligned}$$

Thus, each element of the vector $\dot{\mathbf{s}}^L$ can be written as

$$\begin{aligned}\dot{s}_i^L &= f_i - \hat{f}_i + \sum_{j=1}^{n} \Delta_{ij}\left(\ddot{x}_{dj} - \lambda_j\dot{e}_j - \hat{f}_j\right) \\ &\quad - \sum_{j\neq i}^{n} \Delta_{ij} k_j{\cdot}sgn(s_j) - (1 + \Delta_{ii})\, k_i{\cdot}sgn(s_i).\end{aligned}$$

Therefore, the sliding mode existence condition, $s_i^L \dot{s}_i^L \leq -\eta_i \left|s_i^L\right|$, is verified if

$$\begin{aligned}(1 - \overline{\Delta}_{ii})\, k_i &\geq F_i + \sum_{j=1}^{n} \overline{\Delta}_{ij}\left|\ddot{x}_{dj} - \lambda_j\dot{e}_j - \hat{f}_j\right| \\ &\quad + \sum_{j\neq i}^{n} \overline{\Delta}_{ij} k_j + \eta_i,\end{aligned}$$

where $i = 1, 2, \cdots, n$. Since $\mathbf{k}$ was chosen from (4), it is clear that the above inequality can be guaranteed. Hence, the existence and uniqueness of non-negative k_i's satisfying (4) is guaranteed by Frobenius-Perron theorem [12]. ∎

Theorem 2 (Frobenius-Perron) *[12] Consider a square matrix* $\mathbf{A}$ *with non-negative elements. Then, the largest real eigenvalue of* $\mathbf{A}$*,* λ_{max}*, is non-negative. Furthermore, consider the following equation:*

$$\left(\mathbf{I} - \lambda^{-1}\mathbf{A}\right)\mathbf{y} = \mathbf{z}, \quad (5)$$

where all components of the vector $\mathbf{z}$ *are non-negative. If* $\lambda > \lambda_{max}$*, then the above equation admits a unique solution* $\mathbf{y}$*, whose components* y_i *are all non-negative.*

3.2 The Second Stage

When s_i hits the sliding hyperplane, the linear sliding hyperplane, s_i^L, is substituted by the terminal sliding hyperplane, s_i^{NL}, which will be defined later. Suppose that the hitting state can be represented as

$$s_i^L = \dot{e}_i(t_h) + \Lambda e_i(t_h) = 0,$$

where t_h represents the hitting time. Since $\lambda_i > 0$, the hitting state must be located at II or IV quadrants, i.e.,

$$e_i(t_h)\dot{e}_i(t_h) < 0.$$

The nonlinear terminal sliding hyperplane is designed as

$$\mathbf{s}^{NL} = \dot{\mathbf{e}} + \Gamma\mathbf{e}^p, \quad (6)$$

where $\Gamma = diag\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, $\frac{1}{2} \leq p = \frac{q}{r} < 1$, $q = 2k + 1$, $r = 2m + 1$, $0 \leq k < m$, k and m are integers and predetermined. The γ_i is obtained using the condition that, at the hitting instant,

$$s_i^{NL}(t_h) = s_i^L(t_h) = 0.$$

From the above equation, γ_i is

$$\gamma_i = \lambda_i \left| e_i(t_h) \right|^{1-p}.$$

For the nonlinear terminal sliding hyperplane, the following theorem can be derived.

Theorem 3 *The existence of the sliding mode can be guaranteed if the following controller is applied to the plant (1):*

$$\mathbf{u}^{NL}(t) = \hat{\mathbf{G}}^{-1}\left[\ddot{\mathbf{x}}_{\mathbf{d}} + p\Gamma^2\mathbf{e}^{2p-1} - \hat{\mathbf{f}} - \mathbf{h} \bullet \mathbf{sgn(s)}\right], \tag{7}$$

where $\mathbf{h}\bullet\mathbf{sgn(s)}$ *is the vector of components* $h_i \cdot sgn(s_i)$*, i. e.,*

$$\mathbf{h}\bullet\mathbf{sgn(s)} = \left[h_1 \cdot sgn(s_1),\ h_2 \cdot sgn(s_2),\ \cdots,\ h_n \cdot sgn(s_n)\right],$$

and each positive constant h_i *is chosen such that the following equation is satisfied.*

$$\left(1 - \overline{\Delta}_{ii}\right) h_i - \sum_{j \neq i}^{n} \overline{\Delta}_{ij} h_j = F_i + \sum_{j=1}^{n} \overline{\Delta}_{ij} \left| \ddot{x}_{dj} + p\gamma_j^2 e_j^{2p-1} - \hat{f}_j \right| + \beta_i, \tag{8}$$

where $\beta_i > 0$*, and* $i = 1, 2, \cdots, n$.

Proof Differentiating the nonlinear terminal sliding surface $\mathbf{s}^{NL}$ in (6) with respect to time, one can obtain

$$\dot{s}_i^{NL} = \ddot{x}_i - \ddot{x}_{di} + \gamma_i p e_i^{p-1} \dot{e}_i.$$

Note that the state is on the sliding hyperplane, i.e.,

$$s_i^{NL} = \dot{e}_i + \gamma_i e_i = 0.$$

Thus, $\dot{e}_i$ can be rewritten as

$$\dot{e}_i = -\gamma_i e_i.$$

$$\begin{aligned} \dot{s}_i^{NL} &= \ddot{x}_i - \ddot{x}_{di} - p\gamma_i^2 e_i^{2p-1} \\ &= f_i - \hat{f}_i + \sum_{j=1}^{n} \Delta_{ij} \left(\ddot{x}_{dj} + p\gamma_i^2 e_i^{2p-1} - \hat{f}_j \right) \\ &\quad - \sum_{j \neq i}^{n} \Delta_{ij} h_j \cdot sgn(s_j) - \left(1 + \Delta_{ii}\right) h_i \cdot sgn(s_i). \end{aligned}$$

Therefore, the sliding mode existence condition, $s_i^{NL}\dot{s}_i^{NL} \leq -\beta_i \left| s_i^{NL} \right|$, is verified if

$$\begin{aligned} \left(1 - \overline{\Delta}_{ii}\right) h_i \ \geq\ & F_i + \sum_{j=1}^{n} \overline{\Delta}_{ij} \left| \ddot{x}_{dj} + p\gamma_i^2 e_i^{2p-1} - \hat{f}_j \right| \\ & + \sum_{j \neq i}^{n} \overline{\Delta}_{ij} h_j + \beta_i, \end{aligned}$$

where $i = 1, 2, \cdots, n$. Since $\mathbf{h}$ was chosen from (8), it is clear that the above inequality can be guaranteed. Hence, the existence and uniqueness of non-negative h_i's satisfying (8) is guaranteed by Frobenius-Perron theorem [12]. ■

For the finite time error convergence, we can derive the following theorem

Theorem 4 *The total error convergence time,* t_{CONVi}*, is*

$$t_{CONVi} \leq \frac{|s_i(0)|}{\eta_i} + \frac{1}{\gamma_i(1-p)} \left\{ \frac{|s_i(0)|}{\lambda_i} \right\}^{1-p}. \tag{9}$$

Proof In the first stage, we can conclude that wherever the initial state is located, it hits the linear sliding hyperplane in finite time. From Theorem 1, it is guaranteed that

$$s_i \dot{s}_i \leq -\eta_i \left| s_i \right|.$$

Thus, it is clear that the trajectory of a state, x_i, reaches the linear sliding hyperplane, s_i^L, in finite time smaller than $\frac{|s_i(0)|}{\eta_i}$. In other words, the reaching time t_{REACHi} can be written as

$$t_{REACH_i^L} \leq \frac{|s_i(0)|}{\eta_i}. \tag{10}$$

In the second stage, for the nonlinear terminal sliding hyperplane in (6), it is clear that the relaxation time (the time an initial state gets to zero) can be obtained as [11]

$$t_{REACH_i^{NL}} = \frac{|e_i(t_h)|^{1-p}}{\gamma_i(1-p)}. \tag{11}$$

Furthermore, it is also obvious that $e_i(t_h)$ is bounded as

$$\begin{aligned} |e_i(t_h)| \ &\leq\ \frac{|s_i(0)|}{\lambda_i} && (12) \\ &\leq\ \left| e_i(0) + \frac{\dot{e}_i(0)}{\lambda_i} \right|. && (13) \end{aligned}$$

Therefore, substituting the $|e_i(t_h)|$ in (11) as that of (13), $t_{REACH_i^{NL}}$ can be rewritten as

$$t_{REACH_i^{NL}} \leq \frac{1}{\gamma_i(1-p)} \left\{ \frac{|s_i(0)|}{\lambda_i} \right\}^{1-p}, \tag{14}$$

where $i = 1, 2, \cdots, n$. Thus, by adding up the results for two stage, (10) and (11), the total error convergence time can be obtained as

$$t_{CONVi} \leq \frac{|s_i(0)|}{\eta_i} + \frac{1}{\gamma_i(1-p)} \left\{ \frac{|s_i(0)|}{\lambda_i} \right\}^{1-p}.$$

■

4 Conclusions

The sliding mode control with finite time error convergence has been proposed for nonlinear uncertain systems. By using the two stage control methodology, it was guaranteed that the output tracking error converges to zero in *finite* time and there is *no sinularity.*

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