Bayesian Hierarchical Model with Skewed Elliptical Distribution [1]

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Abstract

Meta-analysis refers to quantitative methods for combining results from independent studies in order to draw overall conclusions. We consider hierarchical models including selection models under a skewed heavy tailed error distribution and it is shown to be useful in such Bayesian meta-analysis. A general class of skewed elliptical distribution is reviewed and developed. These rich class of models combine the information of independent studies, allowing investigation of variability both between and within studies, and weight function. Here we investigate sensitivity of results to unobserved studies by considering a hierarchical selection model and use Markov chain Monte Carlo methods to develop inference for the parameters of interest.

KEYWORDS: Bayesian meta-analysis; Density generator; Elliptical distribution; Gibbs sampler; Hierarchical selection model; Metropolis-Hasting algorithm; Skewed elliptical distribution; Weight function.

1. Introduction

Meta-analysis is a quantitative method for combining results from independent studies and combining information which may be used to evaluate cumulative effectiveness, plan new studies and so on, with wide application in the field of medicine. There are two main problems in meta-analyis. One is that the study effects are heterogeneous and usually account for the random effect or hierarchical models (Morris and Normand, 1992). The other is that meta-analysis may have the publication bias for example, only studies with significant results are observed. When there exists the publication bias, the weight function can be used to account for such bias (Larose and Dey, 1997).

Morris and Normand(1992) consider a hierarchical model as follows; for $i = 1, \ldots, n$,

$$\begin{aligned} Y_i|\alpha_i, \sigma_i^2 &\sim N(\alpha_i, \sigma_i^2), \\ \alpha_i|\mu, \sigma_\alpha^2 &\sim N(\mu, \sigma_\alpha^2), \\ \mu \sim N(a, b), \quad \sigma_\alpha^2 &\sim IG(c, d), \end{aligned} \tag{1.1}$$

where $IG(c, d)$ represents the inverse gamma distribution with shape parameter c and scale parameter d. For meta-analysis, here we can interprete that Y_i is the observed study effect, α_i is the true study effect, σ_i^2 is the within-study variance, μ is the average study effect

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and σ_α^2 is between-study variance. We fix σ_i because it is usually the standard error of an estimate. Further, μ and σ_α^2 are assumed to have conjugate priors.

In meta-analysis, model (1.1) is often not the case because it is one of the published research which tends to be biased toward statistical significance (Rosenthal, 1979). To solve such problem, Silliman(1997) introduces hierarchical selection models(HSM) which incorporate weight function into the hierarchical model in (1.1). Thus the probability of observing any specific study effect, y_i, is multiplied by some nonnegative function $w(y_i)$. That is, the random variable Y_i is assumed to be observed from the weighted distribution with density

$$f^w(y_i|\alpha_i, \sigma_i) = \frac{w(y_i)\ f(y_i|\alpha_i, \sigma_i)}{C_w}, \tag{1.2}$$

where $f(y_i|\alpha_i, \sigma_i) = N(y_i; \alpha_i, \sigma_i)$ and

$$C_w = \int w(x)\ f(x|\alpha_i, \sigma_i)dx, \tag{1.3}$$

is the normalizing constant. For the choice of weight function $w(y_i)$, see Larose and Dey(1996) and Silliman(1997). Therefore, Silliman(1997)'s HSM is given as following;

$$\begin{aligned} Y_i|\alpha_i, \sigma_i &\sim f^w(y_i|\alpha_i, \sigma_i), \quad \alpha_i|\mu, \sigma_\alpha^2 \sim N(\mu, \sigma_\alpha^2), \\ \mu &\sim N(a, b), \quad \sigma_\alpha^2 \sim IG(c, d), \end{aligned} \tag{1.4}$$

where $f^w(y_i|\alpha_i, \sigma_i)$ is defined in (2.2). Recently, Chung, Dey and Jang(2000) considered semiparameric approach to hierarchical selection model for Bayesian meta-analysis.

In this paper, we consider hierarchical model with skewed elliptical distribution for Bayesian meta analysis. Here, our Bayesian hierarchical model with skew elliptical distribution is considered as follows; for $i = 1, \ldots, n$,

$$\begin{aligned} Y_i &= \alpha_i + \sigma_i \epsilon_i \\ \epsilon_i \sim SE(0, 1; \delta_\epsilon), &\quad \alpha_i \sim N(\mu, \sigma_\alpha^2) \\ \mu \sim N(a, b), &\quad \sigma_\alpha^2 \sim IG(c, d), \delta_\epsilon \sim \pi(\delta_\epsilon) \end{aligned} \tag{1.5}$$

where $SE(0, 1; \delta)$ denotes the skewed elliptical distribution defind in section 2 later on and δ is called skewness parameter and $\pi(\delta_\epsilon)$ denotes the prior density of δ_ϵ. If $\delta = 0$, error has the symmetric distribution. $IG(c, d)$ represents the inverse gamma distribution with shape parameter c and scale parameter d.

Such non-normal disturbance in statistical model has been investigated by several authors for theoretical and applied interest. Especially as the pioneerer of this area, Zellner(1976) considered a Bayes and classical analysis of linear multivariate Student t regression models. This was extended to scale mixture of normal distributions in Jammalamadaka, Tiwari and Chib(1987) and Chib, Tiwari and Jammalamadaka (1988), whereas Osiewalski(1991) and Chib, Osiewalski and Steel(1991) and Osiewalski and Steel(1993) examined a further generalization to the entire class of multivariate elliptical or ellipsoidal densities. Azzalini and Dalla-Valle(1996) present a gneral theory for the multivariate version of skew-normal distribution which extends the class of normal distributions by the addition of a shape parameter. Also Azzalini and Capitanio(1999) demonstrated that this distribution has resonable flexibility in real data fitting. Recently, Branco and Dey(2000) proposed a general class of multivariate skew-elliptical distributions which contains the multivariate normal, Student t, exponential power and Pearson type II, but with an extra parameter to regulate skewness. Sahu and Dey(2000) considered the regression problem under a skew-elliptical error distribution and developed a Bayesian methodology for the inference of regression parameters.

The rest of this article is organized as follows. Section 2 reviews and develops the multivariate skew elliptical distribution. The particular cases of normal and Student t distributions are explained as examples. In section 3, we develop Bayesian hierarchical model with skewed elliptical errors which also contains the hierarchical selection models. Also, we consider the detail computational scheme under skewed normal and Student t distribution using MCMC method.

2. Skewed Elliptical Distributions

In this section, we consider the skewed elliptical distribution which is generalized from the skewed normal distribution proposed by Azzalini and Dalla-Valle(1996).

2.1. Multivariate elliptical distribution

Definition 2.1. An $n \times 1$ random vector Y is said to have an elliptical distribution with parameters μ(teh location vector) and Σ(dispersion matrix) of dimensions $n \times 1$ and $n \times n$, respectively, with Σ being positive definite if Y has density function of the form

$$f_Y(y) = |\Sigma|^{-1/2} h^{(n)}[(y-\mu)^t \Sigma^{-1}(y-\mu)], \tag{2.1}$$

for a non-increasing function $h^{(n)}, u \geq 0$, such that

$$h^{(n)}(u) = \frac{\Gamma(n/2)}{\pi^{n/2}} \frac{h(u;n)}{\int_0^\infty u^{n/2-1} h(u;n) du}, \tag{2.2}$$

where $h(u)$ is a non-decreasing function such that the integral $\int_0^\infty u^{n/2-1}h(u;n)du$ exists. In this paper we shall always assume that the existence of the density (2.4). The function $h^{(n)}$ is called the density generator and we write $Y \sim El_n(\mu, \Sigma; h^{(n)})$. Thus, $Z = \Sigma^{-1/2}(Y-\mu) \sim El_n(0, I_n; h^{(n)})$ and has a spherical density function $h^{(n)}(||z||^2)$, $z \in R^n$.

2.2. Chen, Dey and Shao's Method

2.2.1. Univariate skew elliptical distribution

In this section we present a class of skewed elliptical diatribution using the approach given in Chen, Dey and Shao(1999), where the skewed random variable evolve from a sum of a symmetric and positive random variables, and is given as

$$Y = \delta Z + \epsilon. \tag{2.3}$$

The important point here is to have ϵ having a symmetric, unimodal and Z having a positive skewed distribution. When $\delta = 0$ we get the original symmetric distribution. The parameter δ has as easy interpretation and is called the skewness parameter, when $\delta > 0 (\delta < 0)$ means right(left) skewness. To see this, we assume that Z and ϵ have upto third moments. Let σ_Z^2 and σ_ϵ^2 be tha variances of Z and ϵ, respectively, and μ_Z^3 be the standardized third moment of Z, i.e. $\mu_Z^3 = E[(Z_E(Z))/\sigma_Z]^3$. Then, the standardized third moment of Y is given by

$$\mu_Y^3 = E[\frac{Y_E(Y)}{\sigma_Y}]^3 = \frac{\delta^3\sigma_Z^3\mu_Z^3}{\sigma_Y^3}, \tag{2.4}$$

where $\sigma_Y^2 = Var(Y) = \delta^2\sigma_Z^2 + \sigma_\epsilon^2$. We are assuming that Z is skewed to the right, therefore $\mu_Z^3 > 0$. Then (2.9) implies that the marginal distribution of Y is skewed to the right(left) when $\delta > 0(\delta < 0)$.

From now, we expand the results to the simple elliptical class, where $\epsilon \sim EL(0, 1, g_2)$ and $z \sim El^+(0, 1, g_1)$, a positive elliptical distribution.

Theorem 2.1. (Branco and Dey, 1999) Suppose there is a generator function g such that

$$g_2(u) = \frac{\pi^{1/2}}{\Gamma(1/2)} \int_0^\infty g(r + u)dr \tag{2.5}$$

and

$$g_1(u) = \frac{g(u + q(\epsilon))}{g_2(q(\epsilon))}, q(\epsilon) = \epsilon^2, \tag{2.6}$$

where g_2 and g_1 are respectively the generator function of ϵ and Z. Then, the probability density function(pdf) of Y as defined in (2.) is

$$f_Y(y) = 2(\delta^2 + 1)^{-\frac{1}{2}} f_{g_2}[\frac{y}{(\delta^2 + 1)^{\frac{1}{2}}}] F_{g_q(y)}[\frac{y\delta}{(\delta^2 + 1)^{\frac{1}{2}}}] \tag{2.7}$$

where f_g and F_g are respectively the pdf and the cdf(cumulative distribution function) of $El(0, 1; g)$ and

$$g_q(y)(u) = \frac{g(u + q(u))}{g_2(q(u))}, q(u) = u^2. \tag{2.8}$$

2.2.2. Conditioning approach to multivariate case

Let ϵ and Z denote m dimensional random vectors. Let μ be an m dimensional vector and Σ be $m \times m$ positive definite matrix. Assume

$$X = (\epsilon, Z)^t \sim El_{2m}(\theta, \Omega; g^{(2m)}), \tag{2.9}$$

where

$$\theta = (\mu, 0)^t, \quad \Omega = \begin{pmatrix} \Sigma & 0 \\ 0 & I \end{pmatrix}$$

and I is the $m \times m$ identity matrix. We consider a skew elliptical class of distributions by using the tranformation

$$Y = DZ + \epsilon \tag{2.10}$$

where D is a diagonal matrix with elements $\delta_1, \ldots, \delta_m$. Let $\delta = (\delta_1, \ldots, \delta_m)^t$. The class is developed by considering the random variable

$$[Y|Z > 0] \tag{2.11}$$

where $Z > 0$ means that $Z_i > 0$ for $i = 1, \ldots, m$.

Theorem 2.2. Under the above assumption, the pdf of $Y|Z > 0$ is given by

$$\begin{aligned} f(y|\mu, \Sigma, D; g^{(m)}) = \quad & 2^m f_Y(y|\mu, \Sigma + D^2; g^{(m)}) \times \\ & F([I - D(\Sigma + D^2)^{-1}D]^{-\frac{1}{2}} D(\Sigma + D^2)^{-1} y_* | 0, I; g^{(m)}_{q(y_*)}), \end{aligned} \tag{2.12}$$

where

$$g_a^{(m)}(u) = \frac{\Gamma(\frac{m}{2})}{\pi^{\frac{m}{2}}} \frac{g(a+u; 2m)}{\int_0^\infty r^{\frac{m}{2}-1} g(a+u; 2m) dr}, \quad a > 0, \tag{2.13}$$

and

$$q(y_*) = y_*^t (\Sigma + D^2)^{-1} y_*, \quad y_* = y - \mu. \tag{2.14}$$

Example 2.1. Skew normal distrbution

Let $h(u; m) = e^{-\frac{u}{2}}$. Then it is easy to show that $g^{(u)} = (2\pi)^{-m/2}$ and $g^{(m)}_{q(y_*)}$ is free of $q(y_*)$. Now, the pdf of the skew normal distribution is given by

$$\begin{aligned} f(y|\mu, \Sigma, D) \quad = \quad & 2^m |\Sigma + D^2|^{-1/2} phi_m[(\Sigma + D^2)^{-1/2}(y - \mu)] \times \\ & \Phi_m[(I - D(\Sigma + D^2)^{-1}D)^{-1/2} D(\Sigma + D^2)^{-1}(y - \mu)], \end{aligned} \tag{2.15}$$

where ϕ_m and Φ_m denote the density and cdf of m dimensional normal distribution with mean) and covariance matrix identity.

Example 2.2. Skew t distrbution

Let

$$h(u; 2m, \nu) = [1 + \frac{u}{\nu}]^{-\frac{\nu+2m}{2}}. \tag{2.16}$$

Then

$$g_a^{(m)}(u; \nu) = \Gamma(m)[\pi(\nu + m)]^{-m/2} (\frac{\nu + m}{\nu + a})^m (1 + \frac{u}{\nu + m} \frac{\nu + u}{\nu + a})^{1(\nu+2m)/2}. \tag{2.17}$$

Therefore, the density of the multivariate skew t distribution is given by

$$\begin{aligned} f(y|\mu, \Sigma, \nu) \quad = \quad & 2^m t_{m,\nu}(y|\mu, \Sigma + D^2) \times \\ & T_{m,\nu+m}[(\frac{\nu + q(y_*)}{\nu + p})^{-1/2}(I - D(\Sigma + D^2)^{-1}D)^{-1/2} D(\Sigma + D^2)^{-1} y_* \end{aligned} \tag{2.18}$$

3. Bayesian hierarchical model with skewed error

We consider hierarchical model where the error distribution follows the skew elliptical distribution. Suppose that we have n independent observed one-dimensional response variables y_i. That is,

$$Y_i = \alpha_i + \sigma_i \epsilon_i, \quad \epsilon_i \sim SE(0, 1, \delta; g^{(1)}),$$

$$\alpha_i \sim N(\mu, \sigma_\alpha), \tag{3.1}$$

Further $y_i \sim SE(\alpha_i, \sigma_i^2, \delta; g^{(1)})$ independently, for $i = 1, \ldots, n$, To completely specify the Bayesian model, we need to specify prior distributions for all the parameters. Let $\alpha = (\alpha_1, \ldots, \alpha_n)^t$ and $\sigma^2 = (\sigma_1^2, \ldots, \sigma_n^2)^t$. We assume $\mu \sim N(a, b)$. For σ_i^2 and σ_α^2, we asumme gamma priors, $\Gamma(k_1, k_1)$ and $\Gamma(k_2, k_2)$, respectively, where the parametrization has mean 1 and k's are assumed to be a known parameters. When the skewed t models are considered, we need prior distribution for the degree of freedom parameter ν.

Now the posterior density is given by

$$\pi(\mu, \alpha, \sigma^2, \sigma_\alpha^2, \delta, \nu | y_1, \ldots, y_n) \propto \tag{3.2}$$
$$\prod_{i=1}^{n}[SE(y_i|\mu + \alpha_i, \sigma_i^2, \delta; g^{(1)})N(\alpha_i|0, \sigma_\alpha^2)] \quad \times \quad \pi(\mu, \sigma, \sigma_\alpha^2, \delta, \nu)$$

where $\pi(\mu, \sigma, \sigma_\alpha^2, \delta, \nu)$ is the joint prior density of $\mu, \sigma, \sigma_\alpha^2, \delta$ and ν. Note that for the skewed normal models, the distribution of ν is omitted.

3.1. Posterior Propriey

In practice we may experiment with improper prior distributions for μ, σ_i^2 and σ^2. A natural question in such a case whether the full posterior distribution is proper.

Theorem 3.1. Suppose that the priors of δ and ν are proper. Then the posterior in (3.) is proper under the skew normal or skew t model if $n > 1$.

3.2. Hierarchical selection model with skew normal error

Assume that z_i and u_i are distributed to the positive normal and the standardized normal distribution, respectively. Then, our Bayesian hierarchical selection model with skewed normal error can be written as follows; for $i = 1, \cdots, n$,

$$Y_i = \alpha_i + \sigma_i(\delta z_i + u_i), \qquad \begin{cases} z_i \sim N^+(0,1) \\ u_i \sim N(0,1) \end{cases}$$

$$\alpha_i \sim N_\mu, \sigma_\alpha^2),$$

and

$$\mu, \sigma_\alpha, \sigma_i, \delta, \sim \pi(\mu, \sigma_\alpha, \sigma_i, \delta). \tag{3.3}$$

Then, for $i = 1, \cdots, n$, the density function under the weigh function $w(y_i)$ can be viewed as

$$y_i|\alpha_i, \sigma_i, \delta, z_i \sim [C(\alpha_i, \sigma_i, \delta, z_i)]^{-1} w(y_i) N(y_i|\alpha_i + \sigma_i \delta z_i, \sigma_i^2),$$

where $C(\alpha_i, \sigma_i, \delta, z_i) = \int w(y_i) N(y_i|\alpha_i + \sigma_i\delta z_i, \sigma_i^2) dy_i$ is the normalizing constant.

Therefore, the joint density function of y_i, α_i, z_i given $\mu, \sigma_\alpha, \sigma_i, \delta$ is expressed as follows;

$$[y_i, \alpha_i, z_i|\mu, \sigma_\alpha, \sigma_i, \delta] = [y_i|\alpha_i, \sigma_i, \delta, z_i] \quad [\alpha_i|\mu, \sigma_\alpha] \quad [z_i]$$

$$\propto [C(\alpha_i,\sigma_i,\delta,z_i)]^{-1}(\sigma_i^2)^{-\frac{1}{2}}exp\left(-\frac{(y_i-\alpha_i-\sigma_i\delta z_i)^2}{2\sigma_i^2}\right)$$
$$\times(\sigma_\alpha^2)^{-\frac{1}{2}}exp\left(-\frac{(\alpha_i-\mu)^2}{2\sigma_\alpha^2}\right)exp\left(-\frac{{z_i}^2}{2}\right)I(z_i>0,). \tag{3.4}$$

Here, the following priors are assumed;

$$[\mu]\sim N(a,b), \qquad [\delta]\sim N(m,\tau),$$
$$[\sigma_\alpha^2]\sim IG(c_1,d_1)\ , \qquad [\sigma_i^2]\sim IG(c_2,d_2) \tag{3.5}$$

Since the joint posterior density is proportinal to the product of the likelihood function and the prior dendity functions, is given as follows;

$$\prod_{i=1}^{n}\left\{[C(\alpha_i,\sigma_i,\delta,z_i)]^{-1}(\sigma_i)^{-\frac{1}{2}}exp\left(-\frac{(y_i-\alpha_i-\sigma_i\delta z_i)^2}{2\sigma_i^2}\right)\right\}$$
$$\times\prod_{i=1}^{n}\left\{(\sigma_\alpha^2)^{-\frac{1}{2}}exp\left(-\frac{(\alpha_i-\mu)^2}{2\sigma_\alpha^2}\right)exp\left(-\frac{z_i^2}{2}\right)\right\}exp\left(-\frac{(\mu-a)^2}{2b}\right)exp\left(-\frac{(\delta-m)^2}{2\tau}\right)$$
$$\times\prod_{i=1}^{n}\left\{(\sigma_i^2)^{-(c_2+1)}exp\left(-\frac{d_2}{\sigma_i^2}\right)\right\}(\sigma_\alpha^2)^{-(c_1+1)}exp\left(-\frac{d_1}{\sigma_\alpha^2}\right). \tag{3.6}$$

Then in order to apply the Markov chain Monte Carlo(MCMC) method to (3.6), the following full conditional distribution are needed;

$$[\alpha_i|\alpha_j(j\neq i),\underline{y},\underline{z},\mu,\delta,\sigma_\alpha,\underline{\sigma}]$$
$$\sim[C(\alpha_i,\sigma_i,\delta,z_i)]^{-1}N\left(\frac{(\sigma_\alpha^2(y_i-\sigma_i\delta z_i)+\sigma_i^2\mu}{\sigma_\alpha^2+\sigma_i^2},\frac{\sigma_\alpha^2\sigma_i^2}{\sigma_\alpha^2+\sigma_i^2}\right) \tag{3.7}$$

$$[\mu|\underline{\alpha},\underline{y},\underline{z},\delta,\sigma_\alpha,\underline{\sigma}]\sim N\left(\frac{b\sum\alpha_i+\sigma_\alpha^2 a}{bn+\sigma_\alpha^2},\frac{b\sigma_\alpha^2}{bn+\sigma_\alpha^2}\right) \tag{3.8}$$

$$[\delta|\underline{\alpha},\underline{y},\mu,\underline{z},\sigma_\alpha,\underline{\sigma}]\sim[C(\alpha_i,\sigma_i,\delta,z_i)]^{-1}N\left(\frac{\sum\frac{z_i(y_i-\alpha_i)}{\sigma_i}+\frac{m}{\tau}}{\sum z_i^2+\frac{1}{\tau_1}},\left(\sum z_i^2+\frac{1}{\tau}\right)^{-1}\right), \tag{3.9}$$

$$[z_i|\underline{\alpha},\underline{y},\mu,z_j(j\neq i),\delta,\sigma_\alpha,\underline{\sigma}]\sim[C(\alpha_i,\sigma_i,\delta,z_i)]^{-1}N^{+}\left(\frac{\delta(y_i-\alpha_i)}{\sigma_i(1+\delta^2)},(1+\delta^2)^{-1}\right) \tag{3.10}$$

$$[\sigma_\alpha^2|\underline{\alpha},\underline{y},\mu,\underline{z},\delta,\underline{\sigma}]\sim IG\left(n+c_1,\frac{1}{2}\sum(\alpha_i-\mu)^2+d_1\right), \tag{3.11}$$

$$[\sigma_i^2|\underline{\alpha},\underline{y},\mu,\underline{z},\delta,\sigma_\alpha,\sigma_j(j\neq i)] \sim [C(\alpha_i,\sigma_i,\delta,z_i)]^{-1}exp\left(\frac{\delta z_i(y_i-\alpha_i)}{\sigma_i}\right)$$
$$IG\left(\frac{1}{2}+c_2,\frac{1}{2}(y_i-\alpha_i)^2+d_2\right). \tag{3.12}$$

So the sampling for $(\alpha_i,\delta,z_i,\sigma_i)$ is needed the Metropolis-Hastings algorithm (Metropolis *et al.*, 1953; Hastings, 1970).

References

1. Azzalini, A. and Dalla-Valle, A.(1996) The Multivariate Skew-Normal Distribution, *Biometrika*, 83, 15-726.

2. Branco D. and Dey, D.K.(2000) A general class of multivariate skew-elliptical distribution, *Journal of Multivariate Analysis.*

3. Chen, M. H., Dey, D. K. and Shao, Q. M.(1999) A New Skewed Link Model for Dicotomous Quantal Response Data, *Journal of the American Statistical Association*, 94, 1172-1186.

4. Chung, Y., Dey, D.K. and Jang, J.(2000) Semiparametric hierarchical selection model for Bayesian meta analysis, Technical Report 0009, Department of Statistics, University of Connecticut.

5. Dickey, P. and C.-H. Chen(1985) Direct subjective probability modelling using ellipsoidal distributions, in J.M. Bernardo. M.H. DeGroot, D.V. Lindley, and A.F.M. Smith, eds., *Bayesian Statistics* 2, 157-182.

6. Fernandez, C., Osiewalski, J. and Steel, M. F. J.(1995) Modeling and inference with v-spherical distributions, *Journal of Amarican Statistical Association*, 90, 1331-1339.

7. Fernandez, C. and Steel, M. F. J.(1998) On Bayesian Modeling of Fat Tails and Skewness, *Journal of the American Statistical Association*, 93, 359-371.

8. Kelker, D.(1970) Distribution theory of spherical distributions and location-scale generalization, *Sankhya A*, 32, 419-430.

9. Larose, D. and Dey, D. K.(1996) Weighted Distributions Viewed in the Context of Model Selection: A Bayesian Perspective, *Test*, 5, 227-246.

10. Morris, C. N. and Normand, S. L.(1992) Hierarchical Models for Combining Information and for Meta-Analysis, in *Bayesian Statistics* 4, (Bernardo, J. M., Berger, J. O., Dawid, A. P. and Smith, A. F. M. eds.), Oxford, UK; Oxford University Press, pp. 321-344.

11. Osiewalski, J. and Steel, M. F. J.(1993) Robust Bayesian Inference in Elliptical Regression Models, *Journal of Econometrics*, 57, 345-363.

12. Silliman, N. P.(1997) Hierarchical Selection Models with Applications in Meta-Analysis, *Journal of the American Statistical Association*, 92, 926-936.

13. Zellner, A.(1976) Bayesian and Non-Bayesian Analysis of the Regression Model with Multivariate Student-t Error Terms, *Journal of the American Statistical Association*, 71, 400-405.