Asymptotic Properties of Variance Change-point in the Long-memory Process

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Abstract

It is noted that many econometric time series have long-memory properties. A long-memory process, or strongly dependent process, is characterized by hyperbolic decaying autocorrelations and unbounded spectral density at the origin.. Since the long-memory property can be observed by data obtained from rather a long period, there is some possibility of parameter change in the process. In this paper, we consider the estimation of change-point when there is a change in the variance of a long-memory process. The estimator is based on some reasonable statistic and the consistency is shown using Taqqu's strong reduction theorem

Keywords : change-point, long-memory process, strong reduction theorem, Hermite process

1. Introduction

Since some of the first interests in long-memory processes were appeared in the articles by Hurst(1951, 1956), there have been many works on the behaviors of the processes. As was seen in the examples of Hurst's tidal inflow and outflow data, the first and most well known examples of data with long-memory property are discovered in the areas of natural sciences such as hydrology and climatology. Since 1980's, econometricians have noted that many econometric time series also have long-memory property (Ding and Granger, 1996). One of the characteristics of long-memory process is that the autocorrelations of the original series have the appearance of nonstationarity, while the differenced series appear over-differenced. This phenomenon supports the possibility of existence of some process that is neither consistent with a unit root process nor an I(0) process. For this reason, the long-memory process is also called a fractionally differenced process and becomes a common issue of the econometrics and statistics.

On the other hand, from the facts that many recent long memory processes are found in the econometric data and the long memoriness can be detected only through data obtained from quite a long period, the possibility of change in the characterizing parameters cannot be ignored. Moreover, as we can see in the recent economic crisis of Korea in 1997, the variability is the essential characteristic of many econometric variables. And for that reason, the estimation of variance and the detection of change in volatility become very important.

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2. Long-Memory Process

In this paper, we consider both Gaussian and non-Gaussian long-memory processes. As for the Gaussian long-memory process, we will consider the fractional Gaussian noise and for the expansion to the non-Gaussian case, we will consider the linear process with slowly decaying coefficients.

The fractional Gaussian noise(Samorodnitsky and Taqqu, 1994) is represented as

$$u_i = \frac{1}{A(H)} \int_{-\infty}^{i+1} [(i+1-s)_+^{H-1/2} - (i-s)_+^{H-1/2}] dB(s), \qquad i \in Z$$

where

$$A(H) = \{ \frac{1}{2H} + D(H) \}^{1/2}, \quad D(H) = \int_0^{\infty} \{ (1+s)^{H-1/2} - s^{H-1/2} \}^2 ds$$

and the parameter H is the self-similarity parameter of the limit process of the partial sum process $\sum_1^{[n\tau]} u_i$.

If we consider a linear process $u_i = \sum_{j=0}^{\infty} a_j \varepsilon_{i-j}$ where the error process $\{\varepsilon_i\}$ is *IID* with mean zero and the finite fourth moment and the coefficient sequence $\{a_j\}$ satisfies

$$a_j \sim L(j) j^{H-3/2}, \quad 1/2 < H < 1$$

for some regularly varying function $L(\cdot)$, this linear process has asymptotically same properties as fractional Gaussian noise.

For the test of the parameter change, we will consider the weak convergence and convergence rate of the process $(1/\sqrt{A(n)}) \sum_{1 \le i \le [n\tau]} G(u_i)$ for $0 < \tau < 1$ where $\{u_i\}$ is a long-memory process with parameter H. By Taqqu(1975, 1977)'s strong reduction theorem, Giraitis and Taqqu(1999) shows that the limiting properties of the above form only depends on the Hermite rank of the function G if the process $\{u_i\}$ is Gaussian and the limiting distribution is asymptotically equivalent to

$$\frac{J(m)}{m!} \frac{1}{\sqrt{A(n)}} \sum_{1 \le i \le [n\tau]} H_m(u_i)$$

where m is the Hermite rank of G and $J(m) = EH_m(u)G(u)$. If $\{u_i\}$ is non-Gaussian, the same argument holds for the Appell polynomial $P_m(x)$ and Appell rank.

In detail, if $d_m(H) = 1 - m(2-2H) < 0$ then the process $H_m(u_i)$ is weakly dependent and the limit of the process is Gaussian, but if $d_m(H) = 1 - m(2-2H) > 0$, the process $H_m(u_i)$ is strongly dependent and the limit is non-Gaussian.

'3. Estimation of the Change-Point

Consider the following variance-change model

$$X_i = \begin{cases} \mu + \sigma u_i, & i = 1, \cdots, k^* \\ \mu + \theta u_i, & i = k^* + 1, \cdots, n \end{cases}$$

where the error process is a long-memory process with parameter H and the mean μ is known. Then as in the study of Gombay et al.(1996), we can detect the change in variance by the functional

$$Z_n^* = sup_{1 \le k < n} \frac{1}{\sigma_0} \left(\frac{n}{k(n-k)}\right)^{\gamma} \left[\sum_1^k (X_i - \mu)^2 - \frac{k}{n} \sum_1^n (X_i - \mu)^2\right]$$

where $\sigma_0 = var_{H_0}(X_1)$ and the reasonable estimator of the change-point is given by

$$\hat{k}^* = \min\{k : k = argmax|Z_n(i)|\}$$

where $Z_n(k) = \left(\frac{n}{k(n-k)}\right)^{\gamma} \left(\sum_{1 \le i \le k} (X_i - \mu)^2 - \frac{k}{n} \sum_{1 \le i \le n} (X_i - \mu)^2\right)$ for some $0 < \gamma < 1$.

If we denote that $\delta = \sigma^2 - \theta^2$ and $k^* = [\tau^* n]$, then the consistency of the estimator is verified as follows.

Theorem 3.1

If the error process is a long-memory process with parameter $H \in (3/4, 1)$, then we have

$$P[|\hat{\tau} - \tau^*| > \varepsilon] = \frac{C}{\varepsilon^2} \times \begin{cases} n^{2H-2}, & \text{if } H > \gamma \\ n^{2H-2} \log n, & \text{if } H = \gamma \\ n^{2\gamma-2}, & \text{if } H < \gamma. \end{cases}$$

For the derivation of the limiting distribution, we put a two-sided Rosenblatt process with parameter $H_1 = 2H - 1$ by $Z_2^*(t, H_1) = sign(t) \cdot Z_2(t, H_1)$ and define

$$\xi = argsup_{-\infty < K < \infty}(K^{1/2} Z_2^*(t, H_1) - h(t, \gamma, \tau))$$

where

$$h(t, \gamma, \tau) = \begin{cases} ((1-\gamma)(1-\tau) + \gamma\tau)|t| & , \text{if } t \le \le 0 \\ (\gamma(1-\tau) + (1-\gamma)\tau)|t| & , \text{if } t > 0 \end{cases}$$

and K is a scaling factor satisfying $E_{H_0} K^{-1/2} (X_1 - \mu)^2 = 1$.

We also assume that

if $0 \le \gamma < 2H - 1$, then $\frac{|\delta|}{n^{2H-2} L(n)} \to \infty$ *as* $n \to \infty$ and

if $\gamma = 2H - 1$, then $\frac{|\delta|}{n^{2H-2} L(n) (loglog(n))^{1/2}} \to \infty$ *as* $n \to \infty$.

Moreover, we postulate as well $\sum_{1 \le i \le n} \sum_{1 \le j \le n} \gamma(i - j) = n^{2H} L(n) + O(n^H L^{1/2}(n))$

and that $n^{4H-2}L^{2(n)}$ is smoothly varying and

if $2H-1<\gamma<1$, then $\frac{|\delta|}{n^{\gamma-1}}\to\infty$ *as* $n\to\infty$.

Finally, denote $u(t)=t^{2-2H}L^{-1}(t)$ then $u(t)$ is a regularly varying at infinity and therefore has a generalized inverse $u^{-1}(t)$. If we let $\varDelta=u^{-1}(1/\delta)$ then the following theorem is derived.

Theorem 3.2

If the aboce conditions for some $H\in(3/4,1)$ *and* H_A *hold, then*

$$\frac{(\hat{k}^{*}-k^{*})}{\varDelta}\to_{D}\xi$$

4. Conclusion

When there is a change in variance in a long-memory process, the test statistic used for the IID case can be also applied. We showed the consistency of the test statistic when the error process is a long-memory proces. The limit distribution of the test statistic is dependent on the nuisance parameter H and so, we will give some simulation result to apply the test statistic to real examples. We also show the consistency of the estimator of the change-point based on the test statistic and derive the limiting distribution of the estimator.

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