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STOCHASTIC ORDERS IN RETRIAL QUEUES AND THEIR APPLICATIONS

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Abstract

We consider a Markovian retrial queue with waiting space in which the service rates and retrial rates depend on the number of customers in the service facility and in the orbit, respectively. Each arriving customer from outside or orbit decide either to enter the facility or to join the orbit in Bernoulli manner whose entering probability depend on the number of customers in the service facility. In this paper, a stochastic order relation between two bivariate processes $(C(t), N(t))$ representing the number of customers $C(t)$ in the service facility and $N(t)$ one in the orbit is deduced in terms of corresponding parameters by constructing the equivalent processes on a common probability space. Some applications of the results to the stochastic bounds of the multi-server retrial model are presented.

keywords : multi-server retrial queue; waiting room; stochastic order

1. MODEL DESCRIPTION

We consider a Markovian retrial queueing model with service facility. Customers arrive from outside according to a Poisson process with rate λ. An arriving customer who finds k customers in the service facility enters the facility with probability p_k or joins the orbit with probability $1-p_k$ and retries to get service after exponential time. If returning customer from orbit finds k customers in the service facility, then it enters the facility with probability u_k or joins the orbit again with probability $v_k=1-u_k$ and retries its service after exponential time. Let θ_k be the total retrial rate when $k$customers are in the orbit. When there are n customers at the service facility, we assume that, in the absence of arrivals, the time until the next service completion is exponential with rate μ_n.

We are interested in the bivariate process $X(t)=(C(t), N(t))$ called the *CN-process*, where $C(t)$ and $N(t)$ represent the numbers of customers in the service facility and the orbit, respectively, at time t. Obviously, the bivariate process $\{X(t),\ t\geq 0\}$ is a Markov chain with the lattice set $E=Z^+\times Z^+$ where $Z^+=\{0,1,2,\cdots\}$, as the state space.

In this paper, instead of studying a performance measure in a quantitative fashion, we attempt to investigate a relationship between the sample paths of two *CN-processes* from the relationship between parameters by constructing equivalent processes on a common

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probability space.

Define a relation $<$ on E by $(i,j)<(k,l)$ if and only if $j\leq l$ and $i+j\leq k+l$, then it is immediate that $<$ is a partial order on E. If two stochastic processes Y and X are defined on the same probability space then X is almost surely smaller than Y with respect to $<$, written $X<_{as}Y$

if $P(X(t)<Y(t)$ *for all* $t\geq 0)=1$. Note that if $X<_{as}Y$ and Y and X have weak limits $\hat{X}$ and $\hat{Y}$, respectively, then $\hat{X}<\hat{Y}$ in distribution.

2. MAIN RESULTS

Theorem 1. Let $X^{(i)}=\{X^{(i)}(t),\ t\geq 0\}$, $i=1,2$ be the *CN* processes described in the previous section with arrival rates $\lambda^{(i)}$, service rates $\mu_n^{(i)}$ with $\mu_0^{(i)}=0$, retrial rates $\theta_n^{(i)}$ with $\theta_0^{(i)}=0$ and the probabilities $p_n^{(i)}$ and $u_n^{(i)}$ of entering the service facility of customers from outside and from the orbit, respectively. Suppose that $X^{(i)}$, $i=1,2$ are regular and

(*i*) $X^{(1)}(0)=(i,j)<X^{(2)}(0)=(i',j')$
(*ii*) $\lambda^{(1)}\leq\lambda^{(2)}$
(*iii*) $\mu_n^{(1)}\geq\mu_n^{(2)}$ and $\mu_n^{(i)}\leq\mu_{n+1}^{(i)}$, $i=1,2,\ n=0,1,2,\cdots$,
(*iv*) $\theta_n^{(1)}\geq\theta_n^{(2)}$ $n\geq 0$,
(*v*) $p_n^{(1)}\geq p_n^{(2)}$ and $p_n^{(i)}\geq p_{n+1}^{(i)}$, $i=1,2,\ n=0,1,2,\cdots$,
(*vi*) $u_n^{(1)}\geq u_n^{(2)}$ and $u_n^{(i)}\geq u_{n+1}^{(i)}$, $i=1,2,\ n=0,1,2,\cdots$.

Then there are *CN* processes $\hat{X}^{(i)}$, $i=1,2$ which are equivalent to $X^{(i)}$, $i=1,2$, respectively, on a common probability space such that

$$\hat{X}^{(1)}<_{as}\hat{X}^{(2)}.$$

□

Corollary 1. Let $\Sigma(c,K,M)$ be the retrial queue with parameters $\mu_n=\min(c,n)\mu$, $\theta_n=\min(M,n)\theta$ and $p_n=u_n=1$ for $n\leq K-1$ and $p_n=u_n=0$ for $n\geq K$. Let $\lambda^{(k)}$, $\mu_n^{(k)}$ and $\theta_n^{(k)}$ be the corresponding parameters of $\Sigma(c_k,K_k,M_k)$, $k=1,2$ satisfying $\lambda^{(1)}\leq\lambda^{(2)}$, $\mu^{(1)}\geq\mu^{(2)}$ and $\theta^{(1)}\geq\theta^{(2)}$. Then for $c_1\geq c_2$, $K_1\geq K_2$ and $M_1\geq M_2$ and the initial values (i_k,j_k) of the systems $\Sigma(c_k,K_k,M_k)$, $k=1,2$, satisfying $(i_1,j_1)<(i_2,j_2)$, we have

$$\Sigma(c_1,K_1,M_1)<_{as}\Sigma(c_2,K_2,M_2)$$

where $<_{as}$ means that the corresponding *CN*-processes are related by the partial order $<_{as}$.

□

Corollary 2. Let $\Sigma(c_k,K_k,M_k)$ (with $K\leq M$) be the retrial queue with c_k parallel servers and waiting space K_k including service space in which the intensity of retrial becomes infinity as soon as the number of customers in orbit reaches some level M_k and with arrival rate $\lambda^{(k)}$, service rate $\mu^{(k)}$ of each server and retrial rate $\theta^{(k)}$, $k=1,2$ satisfying $\lambda^{(1)}\leq\lambda^{(2)}$, $\mu^{(1)}\geq\mu^{(2)}$ and $\theta^{(1)}\geq\theta^{(2)}$.

Let (i_k,j_k) be the initial states of the system $\Sigma(c_k,K_k,M_k)$, $k=1,2$. If $c_1\geq c_2$, $K_1\geq K_2$, $M_1\leq M_2$, and $(i_1,j_1)<(i_2,j_2)$, then we have

$$\Sigma(c_1, K_1, M_1) <_{as} \Sigma(c_2, K_2, M_2). \qquad \square$$

Corollary 3. (Comparisons of impatient customers) Let $\Sigma_I^{(i)}(c, K, M_i)$ be retrial queues with impatient customers and with parameters $\lambda^{(i)}$, $\mu^{(i)}$, $\theta^{(i)}$, $\alpha^{(i)}$ and $\beta^{(i)}$, $i=1,2$. Suppose that $\lambda^{(1)} \le \lambda^{(2)}, \mu^{(1)} \ge \mu^{(2)}, \theta^{(1)} \ge \theta^{(2)}, \alpha^{(1)} \le \alpha^{(2)}$ and $\beta^{(1)} \le \beta^{(2)}$. Then for $M_1 \ge M_2$, we have

$$\Sigma_I^{(1)}(c, K, M_1) <_{as} \Sigma_I^{(2)}(c, K, M_2). \qquad \square$$

Corollary 4. (Convergence of stationary distributions) We assume that the stability condition $\rho = \frac{\lambda}{c\mu} < 1$ and let $\widetilde{p}_{ij}^{(M)}$, p_{ij} and $p_{ij}^{(M')}$be the stationary distributions of the CN-processes in $\widetilde{\Sigma}(c, K, M)$, $\Sigma(c, K, \infty)$ and $\Sigma(c, K, M')$, respectively. Then we have the relation

$$\{\widetilde{p}_{ij}^{(M)}\} < \{p_{ij}\} < \{p_{ij}^{(M')}\}, \ M, M' \ge 0$$

and

$$\lim_{M \to \infty} \widetilde{p}_{ij}^{(M)} = p_{ij} = \lim_{M \to \infty} p_{ij}^{(M)}, \ 0 \le i \le K, \ j \ge 0. \qquad \square$$

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