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ON THE INCREMENTS OF (N, d)-GAUSSIAN PROCESSES

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ABSTRACT

In this paper we establish limit results on the increments of (N, d)-Gaussian processes with independent components, via estimating upper bounds of large deviation probabilities on the suprema of (N, d)-Gaussian processes.

Keywords: Gaussian process, regularly varying function, Borel-Cantelli lemma.

1. INTRODUCTION AND RESULTS.

Limit theorems on the increments of Wiener processes and Gaussian processes have been investigated in various directions by many authors [1, 5-8, 11-25]. Furthermore, the moduli of continuity for Gaussian processes and Ornstein-Uhlenbeck processes have been intensively studied recently by Csörgő and Shao [9], Csáki and Csörgő [2], Csáki et al. [3, 4], Csörgő et al. [10].

Throughout this paper we shall always assume the following statements: Let $\{X_i(\mathbf{t}),\ \mathbf{t} \in [0,\infty)^N\}$, $i = 1, 2, \cdots, d$, be real-valued independent continuous and centered Gaussian processes with $X_i(\mathbf{0}) = 0$ and $E\{X_i(\mathbf{t}) - X_i(\mathbf{s})\}^2 = \sigma_i^2(\|\mathbf{t} - \mathbf{s}\|)$, where $\sigma_i(t), t > 0$, are positive nondecreasing continuous functions and $\|\cdot\|$ denotes a usual Euclidean norm such that

$$\|\mathbf{t} - \mathbf{s}\| = \Big\{ \sum_{i=1}^{N} (t_i - s_i)^2 \Big\}^{1/2}, \quad t_i, s_i \in [0, \infty).$$

Further assume that $\sigma_i(\cdot)$ are regularly varying functions with exponents α_i at zero and ∞ for some $0 < \alpha_i < 1$.

Let $X^d(\mathbf{t}) = (X_1(\mathbf{t}), \cdots, X_d(\mathbf{t})) \in R^d$, $\mathbf{t} \in [0,\infty)^N$, be a vector-valued Gaussian process. We call $X^d(\mathbf{t})$ a *(N, d)-Gaussian process with independent components.*

For our purpose we first introduce some notations to be used in this paper. Let $\mathbf{t} = (t_1, \cdots, t_N)$ and $\mathbf{s} = (s_1, \cdots, s_N)$ be vectors in $[0,\infty)^N$. We define:

$$(\mathbf{t}, \mathbf{s}) = (t_1, \cdots, t_N, s_1, \cdots, s_N) \in [0,\infty)^{2N};$$
$$\mathbf{t} \le \mathbf{s} \quad \text{if} \quad t_i \le s_i \quad \text{for all integers} \quad 1 \le i \le N;$$
$$\mathbf{t} \pm \mathbf{s} = (t_1 \pm s_1, \cdots, t_N \pm s_N); \quad b\mathbf{t} = (bt_1, \cdots, bt_N) \quad \text{for} \quad b \in [0,\infty);$$
$$\mathbf{ts} = (t_1 s_1, \cdots, t_N s_N)\ ; \quad \mathbf{T} = (T, \cdots, T) \in (0,\infty)^N;$$
$$\mathbf{a}(\mathrm{T}) = (a_1(T), \cdots, a_N(T)) \quad \text{for} \quad 0 < T < \infty,$$

where $a_i(T)$, $i = 1, 2, \cdots, N$, are real-valued positive continuous functions of T.

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For $i = 1, 2, \cdots, N$, let $a_i(T)$ be such that

(i) $\quad 0 < a_i(T) \le T$ and $\liminf_{T\to\infty} a_i(T) > 0$.

For convenience, we denote:

$$\beta_1(T) = \Big\{2\Big(\log\Big(\prod_{i=1}^{N}\frac{T}{a_i(T)}\Big) + \log\Big(b_T + \frac{1}{b_T}\Big)\Big)\Big\}^{1/2},$$

$$\beta_2(T) = \Big\{2\Big(\log\Big(\prod_{i=1}^{N}\frac{T}{a_i(T)}\Big) + \log\log T\Big)\Big\}^{1/2},$$

$$\beta_3(T) = \Big\{2\Big(\log\Big(\prod_{i=1}^{N}\frac{T}{a_i(T)}\Big) - \log_{(m+2)} T\Big)\Big\}^{1/2}, \qquad m \ge 2,$$

$$\sigma(d,h) = \Big\{\frac{1}{d}\sum_{i=1}^{d}\sigma_i^2(h)\Big\}^{1/2}, \quad h > 0,$$

where $\log_{(m)} T = \log\log_{(m-1)} T$ for $m \ge 2$, $\log_{(1)} T = \log T$, $\log x = ln(x \vee 1)$ and $p \vee q = \max\{p, q\}$.

Our main results are as follows:

Theorem 1.1. *Suppose that*

(ii) $$\lim_{T\to\infty}\Big(\frac{T}{a_i(T)} + b_T + \frac{1}{b_T}\Big) = \infty, \quad i = 1, 2, \cdots, N.$$

Then

$$\liminf_{T\to\infty}\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{T}}\sup_{\mathbf{0}\le\mathbf{s}\le\mathbf{a}(\mathrm{T})}\frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\beta_1(T)} \le 1 \qquad \text{a.s.}$$

If we have

(iii) $$\lim_{T\to\infty}\frac{T/a_i(T)}{\log_{(m)} T} = \infty, \quad i = 1, 2, \cdots, N,$$

where $m \ge 2$, then

$$\liminf_{T\to\infty}\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{T}}\sup_{\mathbf{0}\le\mathbf{s}\le\mathbf{a}(\mathrm{T})}\frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\beta_3(T)} \le 1 \qquad \text{a.s.}$$

Theorem 1.2. *Suppose that*

(iv) $$\lim_{T\to\infty}\frac{\log(T/a_i(T))}{\log(b_T + \frac{1}{b_T})} = \infty, \quad i = 1, 2, \cdots, N$$

and that there exist positive constants c_1 and c_2 such that, for $x > 0$,

$$\Big|\frac{d\sigma_i^2(x)}{dx}\Big| \le c_1\frac{\sigma_i^2(x)}{x} \quad \text{and} \quad \Big|\frac{d^2\sigma_i^2(x)}{dx^2}\Big| \le c_2\frac{\sigma_i^2(x)}{x^2}, \quad i = 1, 2, \cdots, d,$$

then

$$\liminf_{T\to\infty}\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{T}}\sup_{\mathbf{0}\le\mathbf{s}\le\mathbf{a}(\mathrm{T})}\frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\beta_1(T)} \ge 1 \qquad \text{a.s.}$$

Combining Theorems 1.1 and 1.2, we have the following limit theorem:

Corollary 1.1. *Under the assumptions of Theorem 1.2, we have*

$$\liminf_{T\to\infty} \sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{T}} \sup_{\mathbf{0}\le\mathbf{s}\le\mathbf{a}(T)} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\beta_k(T)} = 1, \quad k = 1, 2, 3, \quad \text{a.s.}$$

The proofs of Theorems 1.1 and 1.2 are accomplished by applying the modifications of Fernique's lemma [12], Theorem 4.2.1 in Leadbetter et al. [17] and some supplementary lemmas. The details are omitted.

REFERENCES

[1] Y. K. Choi and N. Kôno (1991), On the asymptotic behavior of Gaussian sequences with stationary increments, J. Math. Kyoto Univ., Vol. 31, No. 3, 643-678.

[2] E. Csáki and M. Csörgő (1992), Inequalities for increments of stochastic processes and moduli of continuity, Ann. Probab., Vol. 20, No. 2, 1031-1052.

[3] E. Csáki, M. Csörgő and Q. M. Shao (1992), Fernique type inequalities and moduli of continuity for l^2-valued Ornstein-Uhlenbeck processes, Ann. Inst. Henri Poincare, Vol. 28, No. 4, 479-517.

[4] E. Csáki, M. Csörgő and Q. M. Shao (1995), Moduli of continuity for l^p-valued Gaussian processes, Acta Sci. Math.(Szeged), Vol. 60, 149-175.

[5] E. Csáki, M. Csörgő, Z. Y. Lin and P. Révész (1991), On infinite series of independent Ornstein-Uhlenbeck processes, Stoch. Proc. Appl., Vol. 39, 25-44.

[6] M. Csörgő and P. Révész (1978), How big are the increments of a multi-parameter Wiener process? Z. Wahrsch. verw. Gebiete, Vol. 42, 1-12.

[7] M. Csörgő and P. Révész (1979), How big are the increments of a Wiener process? Ann. Probab., Vol. 7, 731-737.

[8] M. Csörgő and P. Révész (1981), Strong approximations in probability and statistic, Academic Press, New York.

[9] M. Csörgő and Q. M. Shao (1993), Strong limit theorems for large and small increments of l^p-valued Gaussian processes, Ann. Probab., Vol. 21, No. 4, 1958-1990.

[10] M. Csörgő, Z. Y. Lin and Q. M. Shao (1994), Path properties for l^∞-valued Gaussian processes, Proc. Amer. Math. Soc., Vol. 121, No. 1, 225-236.

[11] M. Csörgő, Z. Y. Lin and Q. M. Shao (1994), Kernel generated two-time parameter Gaussian processes and some of their path properties, Can. J. Math., Vol. 46, No. 1, 81-119.

[12] X. Fernique (1964), Continuité des processus and Gaussiens, C. R. Acad. Sci. Paris, t., Vol. 258, 6058-6060.

[13] K. Grill (1991), Upper classes for the increments of the fractional Wiener process, Probab. Th. Rel. Fields, Vol. 87, 411-416.

[14] D. L. Hanson and R. P. Russo (1983), Some more results on increments of the Wiener process, Ann. Probab., Vol. 11, No. 4, 1009-1015.

[15] D. L. Hanson and R. P. Russo (1989), Some "liminf" results for increments of a Wiener process, Ann. Probab., Vol. 17, No. 3, 1063-1082.

[16] N. Kôno (1996), The exact modulus of continuity for Gaussian processes taking values of a finite dimensional normed space, Trends in Probability and Related Analysis, SAP'96, World Scientific, Singapore, 219-232.

[17] M. R. Leadbetter, G. Lindgren and H. Rootzen (1983), Extremes and Related Properties of Random Sequences and Processes, Springer-Verlag, New York, 81-84.

[18] Z. Y. Lin (1997), How big are the increments of l^p-volued Gaussian processes? Science in China (Series A), Vol. 40, No. 4, 337-349.

[19] Z. Y. Lin and C. R. Lu (1992), Strong limit theorems, Science Press, Kluwer Academic Publishers, Hong Kong.

[20] J. Ortega (1984), On the size of the increments of non-stationary Gaussian processes, Stoch. Process Appl., Vol. 18, 47-56.

[21] J. Ortega and M. Wschebor (1984), On the increments of the Wiener process, Z. Wahrsch. verw. Gebiete, Vol. 65, 329-339.

[22] W. J. Park (1974), On Strassen's version of the law of the iterated logarithm for the two-parameter Gaussian process, J. Multi. Analysis, Vol. 4, 479-485.

[23] P. Révész (1982), On the increments of Wiener and related processes, Ann. Probab., Vol. 10, No. 3, 613-622.

[24] J. Steinebach (1983), On the increments of partial sum processes with multidimensional indices, Z. Wahrsch. verw. Gebiete, Vol. 63, 59-70.

[25] L. X. Zhang (1996), Two different kinds of liminfs on the LIL for two-parameter Wiener processes, Stoch. Proc. Appl., Vol. 63, 175-188.

[26] L. X. Zhang (1996), Some liminf results on increments of fractional Brownian motion, Acta Math. Hungar., Vol. 71, No. 3, 215-240.