Likelihood Ratio Test for the Equality of Two Order Restricted Normal Mean Vectors

Hyojin Jeon[1] and Sungsub Choi[2]

ABSTRACT

In the study of the isotonic regression problem, several procedures for testing the homogeneity of a normal mean vector versus order restricted alternatives have been proposed since Barlow's trial(1972). In this paper, we consider the problem of testing the equality of two order restricted normal mean vectors based on the likelihood ratio principle.

Keywords: Isotonic regression; Likelihood ratio test; Pool-Adjacent-Violator algorithm

1. Introduction

Let X_i and Y_i, $i = 1, 2, \ldots k$, be two independent sets of independent normal random variables with means μ_i and $\theta_i = \mu_i + d$, respectively, and common variance σ^2. We consider testing the equality of two mean vectors under the assumption that μ_i's are *isotonic*; $\mu_1 \leq \cdots \leq \mu_k$. We approach the problem of testing $H_0 : d = 0$ versus $H_1 : d \neq 0$ based on the likelihood ratio principle.

In Section 2, we obtain maximum likelihood estimates (MLEs) of parameters following the standard likelihood theories and derive the likelihood ratio test (LRT). We also obtain the distributions of test statistics. However, we fail to derive the exact distributions of test statistics when the variance is unknown. Instead, we propose an F-like test statistic and examine its distribution through a simulation study using the Pool-Adjacent-Violators algorithm in Section 3. Finally, in Section 4, we suggest some subject forfuture studies.

[1]Shinhan Research Institute, Seoul, 100-102, Korea

[2]Department of Mathematics, Pohang University of Science & Technology, Pohang, 790-784

2. LRT for the Equality of Two Ordered Mean Vectors

Consider a two-parameter family of densities with respect to some measure which need not be specified and may depend on the parameter h but not on the parameter θ:

$$f(y:\theta,h)=\exp[\{\Phi(\theta)+(y-\theta)\varphi(\theta)\}h] \tag{2.1}$$

where Φ is strictly convex and φ is its derivative from the right, θ ranges over an interval of real numbers, and h is positive.

We first present a theorem that plays a key role in our problem:

Theorem 2.1. *Consider a family of distributions of the form (2.1) with $\theta = \mu_i$, isotonic, and $h=\sigma_i^{-2}$, $i=1,\dots,k$, and suppose that independent random samples are taken from these distribution with size m_i and n_i, respectively. Then the isotonic MLEs of μ_i's are furnished uniquely by the isotonic regression of sample means, with weight $(m_i+n_i)\sigma_i^{-2}$.*

Proof. Denote the sample values as x_{ij}, $j=1,\dots,m_i$, and y_{ij}, $j=1,\dots,n_i$, $i=1,\dots,k$, respectively. Let also denote sample means as $\bar{x}_i=\sum_{j=1}^{m_i}x_{ij}/m_i$ and $\bar{y}_i=\sum_{j=1}^{n_i}x_{ij}/n_i$ for each i. Then the pooled log-likelihood function is

$$l(\mu,\sigma)=\sum_i\left[\Phi(\mu_i)+\{\frac{m_i\bar{x}_i+n_i\bar{y}_i}{m_i+n_i}-\mu_i\}\varphi(\mu_i)\right](n_i+m_i)\sigma_i^{-2}$$

By Theorem (1.10) in [3], this is uniquely maximized in the class of isotonic μ's by the isotonic regression of $(m_i\bar{x}_i+n_i\bar{y}_i)/(m_i+n_i)$ with weights $(m_i+n_i)\sigma_i^{-2}$. (See [3] for the discussion of isotonic regression.)

Suppose that two independent random samples of size m_i and n_i were taken from normal distributions with mean μ_i and μ_i+d, respectively, and common variance σ^2 for each $i=1,\dots,k$. The log-likelihood function is then

$$\begin{aligned}l(\mu,d,\sigma)=&-\sum_{i=1}^{k}(m_i+n_i)\log\sqrt{2\pi\sigma^2}\\&-\frac{1}{2\sigma^2}\sum_{i=1}^{k}\left\{\sum_{j=1}^{m_i}(x_{ij}-\mu_i)^2+\sum_{j=1}^{n_i}(y_{ij}-\mu_i-d)^2\right\}\end{aligned} \tag{2.2}$$

Under H_0, we obtain the MLE of μ as $\hat{\mu}^*$, where $\hat{\mu}_1^* \leq \cdots \leq \hat{\mu}_k^*$ is the isotonic regression of pooled sample means $(m_i\bar{x}_i + n_i\bar{y}_i)/(m_i + n_i)$. The MLE of σ^2 is

$$\hat{\sigma}^2 = \{\sum_{i=1}^{k}(m_i + n_i)\}^{-1}\sum_{i=1}^{k}\left\{\sum_{j=1}^{m_i}(x_{ij} - \hat{\mu}_i{}^*)^2 + \sum_{j=1}^{n_i}(y_{ij} - \hat{\mu}_i^*)^2\right\}.$$

On the other hand, under the alternatives, we obtain $\breve{\mu}_i{}^* = \hat{\mu}_i^* - q_i(\bar{y} - \bar{x})$, $i = 1, \ldots, k$, and $\breve{d} = \bar{y} - \bar{x}$, where $q_i = n_i/(m_i + n_i)$, $\bar{x} = \sum_{i=1}^{k} w_i\bar{x}_i / \sum_{i=1}^{k} w_i$, $\bar{y} = \sum_{i=1}^{k} w_i\bar{y}_i / \sum_{i=1}^{k} w_i$, and $w_i = m_i n_i/(m_i + n_i)$. The MLE of σ^2 is

$$\breve{\sigma}^2 = \{\sum_{i=1}^{k}(m_i + n_i)\}^{-1}\sum_{i=1}^{k}\left\{\sum_{j=1}^{m_i}(x_{ij} - \hat{\mu}_i^* + p_i\breve{d})^2 + \sum_{j=1}^{n_i}(y_{ij} - \hat{\mu}_i^* - q_i\breve{d})^2\right\},$$

where $q_i = m_i/(m_i + n_i)$, and it follows that the LRT rejects H_0 for large value of the statistic $-2\log\lambda = \sum_{i=1}^{k}(m_i + n_i)\log(\hat{\sigma}^2/\breve{\sigma}^2)$. Since it is difficult to obtain the exact distribution of this test statistic, we instead propose an approximate test statistic. When σ^2 is known, the LRT rejects H_0 for large values of $T = \sum_{i=1}^{k} \frac{m_i n_i}{m_i + n_i}(\bar{y} - \bar{x})^2/\sigma^2$. We propose to replace σ^2 by $\breve{\sigma}^2$ and compare the resulting F-like test statistic with the F distribution with 1 and $\sum_{i=1}^{k}(m_i + n_i - 1)$ degrees of freedom.

3. Simulation Study

We considered the case of common sample sizes. Let X_i and Y_i be random sample of size $m_i = n_i = n$ from $N(is, \sigma^2)$ and $N(is + d, \sigma^2)$, $i = 1, \ldots, k$, respectively. We chose $\sigma = 0.5$, $k = 5$, $s = 0,\ 0.1,\ 0,25,\ 0.5,\ 1$, $d = 0,\ 0.25,\ 0.5$, and $n =$ 1, 2, 5, 10, 20. Each combination was simulated 1,000 times. We use the Pool-Adjacent-Violator algorithm for calculating the isotonic MLE's. The isotonic regression partitions $1, \ldots, k$ into sets on which $\hat{\mu}^*$ is constant, i.e. into level sets for $\hat{\mu}^*$, called solution blocks. On each of these solution blocks the value of $\hat{\mu}^*$ is the weighted average of the value of a function over the block. Thus it suffices to find the solution blocks, i.e. sets of consecutive elements on each of which $\hat{\mu}^*$ assumes a particular value. See[3] for the description and examples of the algorithm.

		s				
d	n	0	0.1	0.25	0.5	1
	1	1.0991	1.5482	1.8095	2.1948	2.2069
	2	1.0147	1.1853	1.1252	1.2699	1.2441
0	5	1.0013	1.0307	0.9588	1.1087	1.1409
	10	0.9371	1.1041	0.9930	1.0573	1.1143
	20	1.0262	1.0528	1.0516	1.0014	1.0095
	1	1.8084	2.2827	2.6058	4.0465	3.8900
	2	2.3928	2.6298	2.6457	2.8358	2.7797
0.25	5	4.1063	4.3506	4.3911	14.5520	4.3359
	10	7.4292	7.3227	7.3000	7.5306	7.3804
	20	13.4825	14.0139	51.9060	13.7368	14.0139
	1	3.8755	4.8347	6.0662	6.5242	8.0809
	2	6.2660	6.7746	7.3332	7.5024	7.7787
0.5	5	13.5233	13.7535	14.7632	14.3903	14.1890
	10	26.9054	27.1541	26.9192	26.7941	27.1368
	20	40.5493	51.7011	51.9060	51.9450	52.0363

Table 3.1: Empirical means of F-values

By studying empirical distribution of the proposed test statistic, as summarized in part in Table 3.1 and illustrated in Figure 3.1, we were able to acsertain several facts such as the consistency of MLE's, the independence of S^2 and $\check{d}$ (not shown here), and the F-like distribuion of test statistic.

4. Discussion and Future Studies

A multivariare version of isotonic regression is well known [5], and it is easy to derive the test statistic using the likelihood ratio principle when variance is known. Let X_i and Y_i, $i = 1, \ldots, k$, denote a multivariate normal vector (of dimension p) with mean vectors μ_i and $\mu_i + d$, respectively, and common variance-covariance matrix Σ. Recall that the k population means constitute a $p \times k$

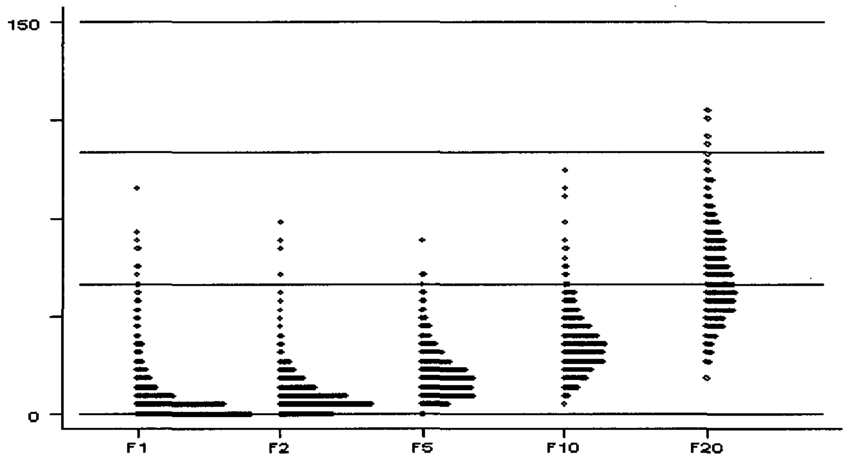

Figure 3.1: The empirical distribution of $F(s = 0.5,\ \ d = 0.5)$

matrices whose rows are isotonic.

Consider testing $H_0:\ d = 0$ versus $H_1:\ d \neq 0$. The log-likelihood is

$$\begin{aligned} l(\mu, d) &= -kp\log(2\pi) - \log|\Sigma| \\ &\quad -\frac{1}{2}\sum_{i=1}^{k}\left\{(X_i-\mu_i)^T\Sigma^{-1}(X_i-\mu_i) + (y_i-\mu_i-d)^T\Sigma^{-1}(Y_i-\mu_i-d)\right\} \quad (4.1) \end{aligned}$$

Under H_0, we obtain the MLE $\hat{\mu}^*$ where $\hat{\mu}_1^* \le \cdots \le \hat{\mu}_k^*$ and $\hat{\mu}_i^* = (\frac{X_i+Y_i}{2})^*$. Under H_1, the MLEs are $\hat{d} = \bar{Y} - \bar{X}$, $\check{\mu}_i{}^* = \hat{\mu}_i^* - 2^{-1}(\bar{Y} - \bar{X})$, $i = 1, \ldots, k$, where $\bar{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$ and $\bar{Y} = \frac{1}{k}\sum_{i=1}^{k} Y_i$.

The likelihood ratio test, therefore, reject H_0 for large value of the statistic $-2\log\lambda = k(\bar{Y} - \bar{X})^T\Sigma^{-1}(\bar{Y} - \bar{X})$. It is χ^2-distributed with p degree of freedom under H_0. The distribution of the test statistic, when Σ is replaced by a sample-based quantity, is left for future studies.

REFERENCES

1 R. P. Abelson & J. W. Tukey,1963, "Efficient utilization of non-numerical information in quantitative analysis : general theory and the case of simple order", *Ann.Math.Statist*, **34** , 1347-69.

2 L. D. Atwood, A. F. Wilson, J. E. Bailey-Wilson, J. N. Carruth, R. C. Elson, 1996, "On the distribution of the likelihood ratio test statistic for a mixture of two normal distribution", *Comm.Statist.Smulation. Comput*, **25** , 733-740.

3 R. E. Barlow, D. J. Bartholomew, J. M. Bremner & H. D. Brunk, 1972, *Statistical Inference under Order Restrictions : The Theory and Application of Isotonic Regression*, Wiley, New York.

4 A. Kudô,1963, "A multivariate analogue of the one-sided test" *Biometrika,* **50** , 403-18.

5 S. Sasabuchi, M. Inutsuka & D. D. S. Kulatunga, 1983, "A multivariate verson isotonic rgression", *Biometrika*, **70** ,465-72.

6 W.Schaafsma & L. J. Smid, 1966, "Most stringent somewhere most powerful tests against alternatives restricted by a number of linear inequalities", *Ann.Math.Statist* , **37** , 1161-72.