# 미분구적법을 이용한 곡선보의 내평면 진동분석

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## 1. Introduction

The early investigators into the in-plane vibration of rings were Hoppe [1] and Love [2]. Love [2] improved on Hoppe's theory by allowing for stretching of the ring. Lamb [3] investigated the statics of incomplete ring with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog [4] used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with simply supported or clamped ends and his work was extended by Volterra and Morell [5] for the vibrations of arches having center lines in the form of cycloids, catenaries or parabolas. Archer [6] carried out for a mathematical study of the in-plane inextensional vibrations of an incomplete circular ring of small cross section with the basic equations of motion as given in Love [2] and gave a prescribed time - dependent displacement at the other end for the case of clamped ends. Nelson [7] applied the Rayleigh-Ritz method in conjunction with Lagrangian multipliers to the case of a circular ring segment having simply supported ends. Auciello and De Rosa [8] reviewed the free vibrations of circular arches and briefly illustrated a number of other approaches. Ojalvo [9] obtained the equations governing three-dimensional linear motions of elastic rings and results for generalized loadings and viscous damping making use of usual classical beam-theory assumptions for the clamped ends. Rodgers and Warner [10] calculated the frequencies of curved elastic rods with simply supported ends.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti [11]. This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. This method is used in the present work to analyze the free in-plane extensional vibrations of curved beams with various boundary conditions and opening angles. The lowest frequencies are calculated for the member. The curved beams considered are of uniform cross section and mass per unit of length and are either clamped or simply supported at both ends. Numerical results are compared with other numerical solutions by a combination of a Holzer-type iterative procedure and an initial value integration procedure.

## 2. System and Governing Equations

The uniform curved beam considered is shown in Fig. 1. A point on the centroidal axis is defined by the angle $\theta$, measured from the left support. The tangential and radial displacements of the arch axis are $v$ and $w$, respectively. $a$ is the radius of the centroidal axis.

Veletsos et al. [12] used a theory which accurately considers the extensibility of the arch axis and the curved beam effect but neglects the effects of rotatory inertia and shearing deformation; they considered simply supported and clamped ends. The differential equations for free vibration of the system which consider the extensibility of the arch axis but neglect the effects of rotatory inertia and shearing deformation, obtained by specializing Flugge's equations for cylindrical shells [13] and incorporating the radial and tangential inertia effects, are

$$w'''' + 2\theta_0^2 w'' + [\theta_0^4 + \theta_0^2(\frac{S}{r})^2 - \lambda^2]w + \theta_0(\frac{S}{r})^2 v' = 0 \qquad (1)$$

$$v'' + \lambda^2(\frac{r}{S})^2 v + \theta_0 w' = 0 \qquad (2)$$

where $S = a\theta_0$ is the length of the arch axis, r is the radius of gyration of the cross section, and $\lambda$ is a dimensionless parameter, related to the circular frequency of vibration of the system, $\omega$, by

$$w=\frac{\lambda}{S^2}\sqrt{\frac{EI}{m}} \tag{3}$$

Here, $m$ is the mass per unit length, $\theta_0$ is the opening angle, $E$ is the Young's modulus of elasticity for the material, and $I$ is the area moment of inertia of the cross section.

Austin and Veletsos [14] studied the free vibrational characteristics of circular arches vibrating in their own planes and presented a simple approximate procedure for estimating the natural frequencies of the systems based on a theory which includes the effects of rotatory inerita.

The differential equations including the effects of rotatory inertia but neglecting shearing deformation, obtained from Federhofer's system [15] are

$$w'''' + [2\theta_0^2 + \lambda^2(\frac{r}{S})^2]w'' + [\theta_0^4 + \theta_0^2(\frac{S}{r})^2 - \lambda^2]w +$$

$$[(\frac{S}{r})^2 - \lambda^2(\frac{r}{S})]\theta_0 v' = 0 \tag{4}$$

$$v'' + [\lambda^2(\frac{r}{S})^2 + \theta_0^2\lambda^2(\frac{r}{S})^4]v + \theta_0[1 - \lambda^2(\frac{r}{S})^4]w' = 0 \tag{5}$$

The boundary conditions for simply supported and clamped ends are, respectively,

$$v = w = w'' = 0 \tag{6}$$

$$v = w = w' = 0 \tag{7}$$

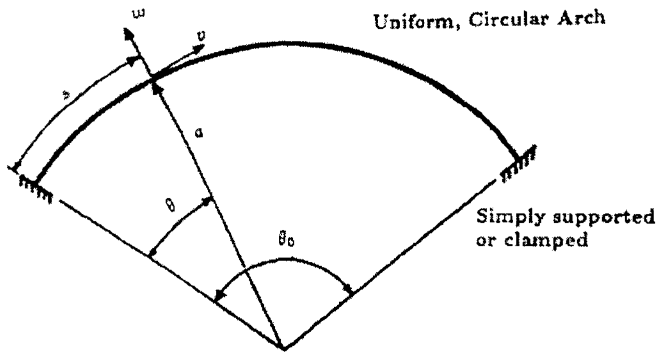


Fig. 1 Arch Considered

## 3. Differential Quadrature Method

The Differential Quadrature Method was introduced by Bellman and Casti [11). By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al. [16)]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Han and Kang [17)] applied the method to the buckling analysis of circular curved beams. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j) \quad \text{for } i, j = 1, 2, \ldots, N \tag{8}$$

where $L$ denotes a differential operator, $x_j$ are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, $W_{ij}$ are the weighting coefficients attached to these function values, and $N$ denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \quad \text{for } k = 1,2,3,\ldots,N \tag{9}$$

If the differential operator $L$ represents an $n^{th}$ derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1} \quad \text{for } i, k = 1,2,\ldots,N \tag{10}$$

This expression represents $N$ sets of $N$ linear algebraic equations, giving a unique solution for the weighting coefficients, $W_{ij}$, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming [18]

## 4. Application

Applying the differential quadrature method to equations (1) and (2), gives

$$\sum_{j=1}^{N} D_{ij}\ w_j + 2\theta^2_0 \sum_{j=1}^{N} B_{ij}\ w_j + [\ \theta^4_0 + \ \theta^2_0 (\frac{S}{r})^2 - \lambda^2] w_i +$$

$$\theta_0 (\frac{S}{r})^2 \sum_{j=1}^{N} A_{ij}\ v_j \ = \ 0 \qquad (11)$$

$$\sum_{j=1}^{N} B_{ij}\ v_j + \lambda^2 (\frac{r}{S})^2 v_i + \theta_0 \sum_{j=1}^{N} A_{ij}\ w_j \ = \ 0 \qquad (12)$$

Similarly, applying the differential quadrature method to equations (4) and (5) gives

$$\sum_{j=1}^{N} D_{ij}\ w_j + [\ 2\theta^2_0 + \lambda^2 (\frac{r}{S})^2] \sum_{j=1}^{N} B_{ij}\ w_j + [\ \theta^4_0 + \ \theta^2_0 (\frac{S}{r})^2 - \lambda^2] w_i +$$

$$[(\frac{S}{r})^2 - \lambda^2 (\frac{r}{S})^2] \sum_{j=1}^{N} A_{ij}\ v_j \ = \ 0$$

$$\sum_{j=1}^{N} B_{ij}\ v_j + [\lambda^2 (\frac{r}{S})^2 + \ \theta^2_0\ \ \lambda^2_0 (\frac{r}{S})^4] v_i +$$

$$\theta_0 [1 - \ \lambda^2_0 (\frac{r}{S})^4] \sum_{j=1}^{N} A_{ij}\ w_j \ = \ 0 \qquad (13)$$

The boundary conditions for simply supported ends, given by equations (6), can be expressed in differential quadrature form as follows:

$$v_1 \ = \ 0 \quad \text{at} \quad X \ = \ 0$$

$$v_N \ = \ 0 \quad \text{at} \quad X \ = \ 1$$

$$w_1 \ = \ 0 \quad \text{at} \quad X \ = \ 0$$

$$\sum_{j=1}^{N} B_{2j}\ w_j = 0 \quad \text{at} \quad X = 0 + \delta$$

$$\sum_{j=1}^{N} B_{(N-1)j}\ w_j = 0 \quad \text{at} \quad X = 1 - \delta$$

$$w_N = 0 \quad \text{at} \quad X = 1$$

Similarly, the boundary conditions for clamped ends, given by equations (7), can be expressed in differential quadrature form as follows:

$$v_1 = 0 \quad \text{at} \quad X = 0$$

$$v_N = 0 \quad \text{at} \quad X = 1$$

$$w_1 = 0 \quad \text{at} \quad X = 0$$

$$\sum_{j=1}^{N} A_{2j}\ w_j = 0 \quad \text{at} \quad X = 0 + \delta$$

$$\sum_{j=1}^{N} A_{(N-1)j}\ w_j = 0 \quad \text{at} \quad X = 1 - \delta$$

$$w_N = 0 \quad \text{at} \quad X = 1$$

## 5. Numerical results and comparisons

The natural frequencies of vibration are calculated by the differential quadrature method and are presented together with results from another method: using a combination of a Holzer-type iterative procedure and an initial value integration procedure by Veletsos et al. [12] and Austin and Veletsos [14]. Here, the values $\lambda$ corresponding to the lowest natural frequencies of vibration neglecting the effects of rotatory inertia and shearing deformation have been evaluated for hinged and fixed arches having angles of opening 45°, 90°, and 180°, for a wide range of the slenderness ratio, S/r. Representative data for this and other cases are tabulated in Tables 1 and 2. The values of $\lambda$ corresponding to the lowest natural frequencies including the effects of rotatory inertia but neglecting shearing deformation have been calculated for hinged and fixed arches $\theta_0 = 90^\circ$ for a wide range of the slenderness ratio, S/r. The results are summarized in Tables 3 and 4. All results are computed with $\delta = 1 \times 10^{-5}$.

Table 1 : Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$ , for simply supported thin arches including mid-surface extension

| S/r | $\theta_0$ | Veletsos al.(1972) | DQM |
|---|---|---|---|
| 11.78 | 90 ° | 18.08 | 18.080 |
| 17.28 | | 25.25 | 25.247 |
| 23.56 | | 33.32 | 33.317 |
| 47.12 | | 33.82 | 33.823 |
| 117.8 | | 33.94 | 33.941 |
| 251.3 | | 33.96 | 33.956 |
| 377.0 | | 33.96 | 33.956 |
| 7.85 | 180 ° | 18.26 | 18.254 |
| 15.71 | | 21.37 | 21.375 |
| 47.12 | | 22.27 | 22.269 |

Table 2 : Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$ , for clamped thin arches including mid-surface extension

| S/r | $\theta_0$ | Veletsos al.(1972) | DQM |
|---|---|---|---|
| 25 | 45 ° | 27.33 | 27.33 |
| 50 | | 39.03 | 39.04 |
| 100 | | 60.08 | 60.108 |
| 12.5 | 90 ° | 26.35 | 26.352 |
| 50 | | 55.37 | 55.326 |
| 100 | | 55.73 | 55.723 |
| 150 | | 55.78 | 55.785 |
| 200 | | 55.81 | 55.803 |
| 250 | | 55.81 | 55.821 |
| 300 | | 55.82 | 55.821 |
| 500 | | 55.84 | 55.830 |

Table 3 : Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$, for simply supported thin arches including mid-surface extension and effects of rotatory inertia ; $\theta_0 = 90°$

| S/r | Austain & Veletsos(1972) | DQM |
|---|---|---|
| 23.56 | 32.55 | 32.542 |
| 47.12 | 33.60 | 33.601 |
| 70.69 | 33.80 | 33.808 |
| 94.25 | 33.87 | 33.867 |
| 141.4 | 33.92 | 33.926 |
| 188.5 | 33.94 | 33.941 |
| 251.3 | 33.95 | 33.956 |
| 314.2 | 33.95 | 33.956 |
| 377.0 | 33.96 | 33.956 |

Table 4 : Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$ , for clamped thin arches including mid-surface extension and effects of rotatory inertia ; $\theta_0 = 90°$

| S/r | Austain & Veletsos(1972) | DQM |
|---|---|---|
| 25 | 37.81 | 37.815 |
| 50 | 54.98 | 54.973 |
| 100 | 55.63 | 55.615 |
| 150 | 55.74 | 55.732 |
| 200 | 55.79 | 55.779 |
| 250 | 55.80 | 55.776 |
| 300 | 55.81 | 55.803 |
| 350 | 55.83 | 55.812 |
| 400 | 55.83 | 55.812 |
| 500 | 55.84 | 55.812 |

## 6. Conclusions

The differential quadrature method was used to compute the eigenvalues of the equations of motion governing the free in-plane extensional vibrations of curved beams. The present method gives results which agree very well with the numerical solutions by other methods for the cases treated while requiring only a limited number of grid points.

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