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# Sequential Test for Parameter Changes in Time Series Models

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ABSTRACT

In this paper, we consider the problem of testing for parameter changes in time series models based on a sequential test. Although the test procedure is well-established for the mean and variance change, a general parameter case has not been discussed in the literature. Therefore, we develop a sequential test for parameter changes in a more general framework.

Key words : Test for parameter changes, sequential test, martingale difference, functional CLT.

## 1 Introduction

The problem of testing for a parameter change has been an important issue among statisticians since the paper of Page (1955). Originally, the problem began with i.i.d. samples; see Hinkley (1971), Brown, Durbin and Evans (1975), Zacks (1983), Csörgő and Horváth (1988), Krishnaiah and Miao (1988) and Inclán and Tiao (1994), and it moved naturally into the time series context since economic time series often exhibit a prominent evidence for structural change in the underlying model; see, for example, Wichern, Miller and Hsu (1976), Picard (1985), Kramer, Ploberger and Alt (1988), Tang and MacNeil (1993), Kim, Cho and Lee (2000), Lee and Park (2001), and the papers cited in those papers.

If the random observations are i.i.d. and follow a parametric model, one may consider utilizing a likelihood ratio method as in Csörgő and Horváth (1997). However, the method is no longer applicable either if the i.i.d. assumption is violated or if the underlying distribution is completely unknown. In such a case, a nonparametric approach should be considered as an alternative. From this viewpoint, here we pay attention to the sequential method for testing for a parameter change.

The sequential method is easy to handle and useful to detect the locations for change points as seen in Inclán and Tiao (1994). However, the usage has been restricted to the change of mean, variance and distribution function (cf. Bai, 1994). A convenience of the method lies in the fact that the sample mean, variance and distribution function are all expressed as the sum of i.i.d. random variables, and the convergence result of the sequential test statistic is easily obtained. The method for a general parameter case has not been

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clarified yet, but an extension is straightforward. We will give a sequential test for changes of general parameters in the next section.

# 2 Main results

Let us consider the test for a mean change in an i.i.d. sample based on the following process:

$$U_n(s) := \frac{1}{\sqrt{n}\sigma}\left(\sum_{t=1}^{[ns]} x_t - \frac{[ns]}{n}\sum_{t=1}^{n} x_t\right) = \frac{[ns]}{\sqrt{n}\sigma}(\hat{\mu}_{[ns]} - \hat{\mu}_n), \quad 0 \le s \le 1, \tag{2.1}$$

where $x_1, \ldots, x_n$ are i.i.d. with mean $\mu$ and variance $\sigma^2$, and $\hat{\mu}_n = n^{-1}\sum_{t=1}^{n} x_t$. It is well-known that $\{U_n\}$ converges weakly to a standard Brownian bridge, and a test is performed based on the convergence result. Similar reasoning can be adopted for the more general case. Suppose that one is interested in testing for a change of $\theta$ based on a consistent estimator $\hat{\theta}_n$. As with the MLE (maximum likelihood estimator), usually $\hat{\theta}_n$ can be written as

$$\hat{\theta}_n - \theta = n^{-1}\sum_{t=1}^{n} l_t + o_P\left(\frac{1}{\sqrt{n}}\right)$$

(cf. Durbin, 1973), where $l_t := l_t(\theta)$ are i.i.d. random variables with zero mean and a second moment. If the $l_t$ are observable as in (2.1), one can construct a sequential test based on

$$\begin{aligned} V_n(s) &:= \frac{1}{\sqrt{n}(El_t^2)^{1/2}}\left(\sum_{t=1}^{[ns]} l_t - \frac{[ns]}{n}\sum_{t=1}^{n} l_t\right) \\ &\simeq \frac{[ns]}{\sqrt{n}(El_t^2)^{1/2}}(\hat{\theta}_{[ns]} - \hat{\theta}_n), \quad 0 \le s \le 1. \end{aligned} \tag{2.2}$$

However, generally the $l_t$ are unobservable, and there must be a justification for having the argument in (2.2). In time series models, $\{l_t\}$ usually forms a sequence of stationary martingale differences.

Now, let us consider the stationary time series $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$, and let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_J)'$ be the parameter vector, which will be examined for constancy, e.g. the mean, variance, autocovariances, etc. Here, we wish to test the following hypotheses based on the estimators $\hat{\boldsymbol{\theta}}_n$:

$H_0$ : $\boldsymbol{\theta}$ does not change for $x_1, \ldots, x_n$. vs.
$H_1$ : not $H_0$.

Let $\hat{\boldsymbol{\theta}}_k$ be the estimator of $\boldsymbol{\theta}$ based on $x_1, \ldots, x_k$. As we saw in (2.2), we investigate the differences $\hat{\boldsymbol{\theta}}_k - \hat{\boldsymbol{\theta}}_n$, $k = 1, \ldots, n$, for constructing a sequential test. The details are addressed below.

Suppose that $\hat{\boldsymbol{\theta}}_k$, obtained from $x_1, \ldots, x_k$, satisfies the following:

$$\sqrt{k}(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}) = \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \boldsymbol{l}_t + \boldsymbol{\Delta}_k, \tag{2.3}$$

where $\boldsymbol{l}_t := \boldsymbol{l}_t(\boldsymbol{\theta}) = (l_{1,t}, \ldots, l_{J,t})'$ forms stationary martingale differences with respect to a filtration $\{\mathcal{F}_t\}$, namely, for every $t$,

$$E(\boldsymbol{l}_t|\mathcal{F}_{t-1}) = \mathbf{0} \ \text{ a.s.}, \tag{2.4}$$

and $\boldsymbol{\Delta}_k = (\Delta_{1,k}, \ldots, \Delta_{J,k})'$.

Let $\Gamma = Var(\boldsymbol{l}_t)$ be the covariance matrix of $\boldsymbol{l}_t$. Assuming that $\Gamma$ is nonsingular, we define the normalized martingale differences

$$\boldsymbol{\xi}_t := \Gamma^{-1/2}\boldsymbol{l}_t.$$

Note that $\boldsymbol{\xi}_t = (\xi_{1,t}, \ldots, \xi_{J,t})'$ has uncorrelated components and satisfies (2.4). Thus if we put

$$\boldsymbol{\xi}_{n,t} = (\xi_{1,n,t}, \ldots, \xi_{J,n,t})' := n^{-1/2}\boldsymbol{\xi}_t,$$

it holds that

$$\sum_{t=1}^{[ns]} \boldsymbol{\xi}_{n,t} \xrightarrow{w} \mathbf{W}_J(s) \tag{2.5}$$

in the $D^J[0,1]$ space (cf. Billingsley, 1968), where $\mathbf{W}_J(s) = (W_1(s), \ldots, W_J(s))'$ denotes a $J$-dimensional standard Brownian motion, since the conditions for functional CLT are satisfied (cf. Gaenssler and Haeusler, 1986):

$$\text{(1) For } j = 1, \ldots, J \text{ and } s \in [0,1], \ \sum_{t=1}^{[ns]} E(\xi_{j,n,t}^2|\mathcal{F}_{t-1}) \xrightarrow{P} s. \tag{2.6}$$

$$\text{(2) For } j = 1, \ldots, J \text{ and } \epsilon > 0, \ \sum_{t=1}^{n} E(\xi_{j,n,t}^2 I(|\xi_{j,n,t}| > \epsilon)|\mathcal{F}_{t-1}) \xrightarrow{P} 0. \tag{2.7}$$

Now, suppose that for each $j = 1, \ldots, J$,

$$\max_{1 \le k \le n} \frac{\sqrt{k}}{\sqrt{n}} |\Delta_{j,k}| = o_P(1). \tag{2.8}$$

Then from (2.3), (2.5) and (2.8), we have that

$$\frac{[ns]}{\sqrt{n}} \Gamma^{-1/2}(\hat{\boldsymbol{\theta}}_{[ns]} - \boldsymbol{\theta}) = \sum_{t=1}^{[ns]} \boldsymbol{\xi}_{n,t} + \Gamma^{-1/2} \frac{\sqrt{[ns]}}{\sqrt{n}} \boldsymbol{\Delta}_{[ns]} \xrightarrow{w} \mathbf{W}_J(s),$$

and consequently,

$$\frac{[ns]}{\sqrt{n}} \Gamma^{-1/2}(\hat{\boldsymbol{\theta}}_{[ns]} - \hat{\boldsymbol{\theta}}_n) \xrightarrow{w} \mathbf{W}_J^{\circ}(s), \tag{2.9}$$

where $\mathbf{W}_J^{\circ}(s) = (W_1^{\circ}(s), \ldots, W_J^{\circ}(s))'$, is a $J$-dimensional standard Brownian bridge. The following is a direct result of (2.3) - (2.9).

**Theorem 2.1** *Define the test statistic* $T_n$ *by*

$$T_n = \max_{1\le j\le J}\max_{1\le k\le n}\left|\frac{k}{\sqrt{n}}\,(\tilde{\theta}_{j,k}-\tilde{\theta}_{j,n})\right|$$

*where* $\tilde{\boldsymbol{\theta}}_k = (\tilde{\theta}_{1,k},\ldots,\tilde{\theta}_{J,k})' := \Gamma^{-1/2}\hat{\boldsymbol{\theta}}_k$. *Suppose that* $\hat{\boldsymbol{\theta}}_k$ *satisfies (2.3) and conditions (2.5) and (2.8) hold. Then, under* $H_0$,

$$T_n \xrightarrow{d} \max_{1\le j\le J}\sup_{0\le s\le 1}|W_j^{\circ}(s)|.$$

*We reject* $H_0$ *if* $T_n$ *is large.*

One can determine the critical region ($T_n \ge C_\alpha$), given a nominal level $\alpha$, where $C_\alpha$ is the $(1-\alpha)$-quantile point of $\max_{1\le j\le J}\sup_{0\le s\le 1}|W_j^{\circ}(s)|$, namely,

$$\alpha = 1-\left(1-P\left(\sup_{0\le s\le 1}|W_1^{\circ}(s)|>C_\alpha\right)\right)^J \tag{2.10}$$

Theorem 2.1 shows that the change point test in time series models can be accomplished based on any estimators provided they satisfy regularity conditions. We can say that the sequential test is widely applicable in a broad class of time series models since it constitutes the most natural nonparametric test, and some well-known estimators, such as the method of moment estimator and the Gaussian MLE, could be employed to perform a test.

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