Finite Element and Boundary Element Modelling of the Acoustic Wave Transmission in Mean Flow Medium

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ABSTRACT

Acoustic field in steady-state is characterized by Helmholtz equation. The transmission characteristics of acoustic wave devices is however influenced by the presence of the mean flow in the medium. The effect of the mean flow introduces additional terms in the equation. In the present paper, two approaches are considered. One is that the equation is directly discretized by FEM for one-dimensional and the axisymmetric case. Another is that the equation is first transformed into the standard Helmholtz equation which is solved by BEM. The numerical demonstrations are made for the axisymmetric FEM and the three-dimensional BEM modeling. The numerical examination for a straight circular duct is first considered. The solutions are compared with the analytical ones. The examination is then extended to the case when the mean flow is locally present in a muffler with expansion chamber.

1 INTRODUCTION

A duct with variable cross-section is the acoustic wave transmission system which has an important field of applications, and its numerical solution has been conducted by many investigators [1]-[6]. Its acoustic wave transmission characteristic is however affected by the presence of the medium motion. The first of all, the propagation velocity may decrease for the waves against the stream while it may increase for the waves along the stream. This is the case when the waves transmit in the exhaust muffler of automobiles.

In the present paper, the formulation is extended to the case of the general impedance termination in one-dimension. The numerical solution by the finite element approach is compared with the analytical solution. The finite element modelling is then applied to the axisymmetric case in which the formulation is made both for the original and transformed coordinates. The boundary element counterpart presented by Zhenlin *et al* [5] is extended to the case of the partial mean flow, which is treated by the partition domain approach. Numerical examples for the duct with an expansion chamber are then demonstrated. All the treatments are made for the steady-state waves in frequency domain. For the time domain wave propagation in mean flow, the present authors proposed the discrete Huygens approach [7].

2 GOVERNING EQUATIONS

We here consider the sound wave propagation in a medium with uniform and steady-state mean flow. The medium is also assumed to be homogeneous and non-dissipative. In the steady-state harmonic motion, the Helmholz equation is given by [7]

$$\nabla^2\Phi + k^2\Phi - j2k(\boldsymbol{M}\cdot\nabla\Phi) - (\boldsymbol{M}\cdot\nabla)(\boldsymbol{M}\cdot\nabla\Phi) = 0 \quad (1)$$

where Φ is the velocity potential, $k = \omega/c_0$ is the wave number, ω is the angular frequency, c_0 is the sound speed at the small amplitude, $\boldsymbol{M} = \boldsymbol{V}_0/c_0$ is the Mach number of the mean flow and $\boldsymbol{V}_0$ is velocity of the mean flow. In the following analysis, the medium is assumed to be moving only in z direction, so that the governing equation is given by

$$\nabla^2\Phi + k^2\Phi - j2kM_z\frac{\partial\Phi}{\partial z} - M_z^2\frac{\partial^2\Phi}{\partial z^2} = 0 \quad (2)$$

where M_z is the mean flow Mach number in z direction.

With the following coordinate and variable transformations,

$$\tilde{x} = x,\ \tilde{y} = y,\ \tilde{z} = \frac{z}{\sqrt{1-M_z^2}},\ \tilde{k} = \frac{k}{\sqrt{1-M_z^2}} \quad (3)$$

$$\tilde{\Phi} = \Phi e^{-jkM_z z} \quad (4)$$

equation (2) leads to the standard Helmholtz equation with respect to $\tilde{\Phi}$ as

$$\tilde{\nabla}^2\tilde{\Phi} + \tilde{k}^2\tilde{\Phi} = 0 \quad (5)$$

where $\tilde{\nabla}^2$ is the Laplacian operator in the transformed coordinates. This expression is numerically solved by Zhenlin et al [5] with the help of the boundary element approach. The equation is simplified at the expense of the complicated boundary conditions. This means that the numerical analysis program such as the finite elements or boundary elements developed for the standard Helmholtz equation can be used without modification, but with the potential and the boundary conditions re-defined.

3 ONE-DIMENSIONAL FIELD

3.1 Analytical solution

Now we consider the wave transmission in the duct as shown in Figure 1. The duct is driven by uniform velocity U_0 at one end ($z = 0$) and it is terminated by the surface acoustic impedance Z_T at another end ($z = \ell$). The boundary conditions for this case correspond to

$$\begin{cases} \frac{\partial\Phi}{\partial n} = -\frac{\partial\Phi}{\partial z} = U_0 & (z=0) \\ \frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial z} = -jk\frac{1}{\beta_T+M_z}\Phi & (z=\ell) \\ \frac{\partial\Phi}{\partial n} = 0 & \text{(wall boundary)} \end{cases} \quad (6)$$

where $\partial/\partial n$ is the normal derivative to the boundary, $\beta_T = Z_T/(\rho_0 c_0)$ is the normalized termination impedance and ρ_0 is the medium density.

For the one-dimensional filed in which the wave propagates toward z direction, the equation (2) is reduced to

$$(1-M_z^2)\frac{\partial^2\Phi}{\partial z^2} + k^2\Phi - j2kM_z\frac{\partial\Phi}{\partial z} = 0 \quad (7)$$

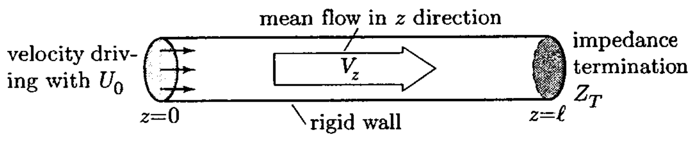

Fig. 1 A duct in mean flow.

The analytical solution for the sound pressure under the boundary condition (6) is

$$P(z) = \frac{-(\rho_0 c_0 - Z_T)e^{-j\frac{k(l-(1+M_z)z)}{1-M_z^2}} + (\rho_0 c_0 + Z_T)e^{j\frac{k(l-(1-M_z)z)}{1-M_z^2}}}{(\rho_0 c_0 - Z_T)e^{-j\frac{kl}{1-M_z^2}} + (\rho_0 c_0 + Z_T)e^{j\frac{kl}{1-M_z^2}}}\rho_0 c_0 U_0 \quad (8)$$

3.2 Finite element formulation

We are to solve equation (7) by the finite element method. The equation is discretized in space by a Galerkin's procedure. The weak formation is

$$\int_\Omega \left\{(1-M_z^2)\frac{\partial\varphi}{\partial z}\frac{\partial\Phi}{\partial z} - k^2\varphi\Phi + j2kM_z\varphi\frac{\partial\Phi}{\partial z}\right\}dz = (1-M_z^2)\varphi\frac{\partial\Phi}{\partial n} \quad (9)$$

where φ is a weighting function. In the Galerkin's method, the weighting function is usually chosen to be the test function ($\varphi = \Phi$). The right hand side of equation (9) corresponds to the boundary condition. For the one-dimensional analysis, the region to be analyzed is divided into line elements. The velocity potential in an element is interpolated by the nodal velocity potentials so that

$$\Phi = \{N\}^T\{\Phi\}_e \quad (10)$$

where $\{\Phi\}_e$ is the nodal velocity potential vector, $\{N\}$ is the interpolation function vector. The superscript ^{T} denotes the transpose of matrix or vector. The velocity potential is here assumed to be linearly interpolated within the line element.

Substituting equation (10) into (9), one has the discretized equation

$$\{(1-M_z^2)[M]_e - k^2[K]_e + jk(2M_z[V]_e + \frac{1-M_z^2}{\beta_T+M_z}[J]_e)\}\{\Phi\}_e = U_0(1-M_z^2)\{W\}_e \quad (11)$$

where $[M]_e, [K]_e, [V]_e$ and $[J]_e$ are inertance, elastance matrices, the matrix associated with mean flow and the wall dissipation and $\{W\}_e$ is the distribution vector. It should be noted that the matrix $[V]_e$ associated with the mean flow is nonsymmetric. The presence of the flow may influence the damping.

4 AXISYMMETRIC FIELD

4.1 Finite element formulation in original coordinates

For the axisymmetric filed, the equation (2) can be expressed as

$$\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+(1-M_z^2)\frac{\partial}{\partial z}\left(r\frac{\partial \Phi}{\partial z}\right)+rk^2\Phi - j2rkM_z\frac{\partial \Phi}{\partial z}=0 \qquad (12)$$

Galerkin's procedure for the axisymmetric filed leads to another weak form as

$$2\pi\iint_\Omega \left\{r\frac{\partial \varphi}{\partial r}\frac{\partial \Phi}{\partial r}+r(1-M_z^2)\frac{\partial \varphi}{\partial z}\frac{\partial \Phi}{\partial z}-rk^2\varphi\Phi + j2rkM_z\varphi\frac{\partial \Phi}{\partial z}\right\}drdz = 2\pi\int_\Gamma r(1-M_z^2)\varphi\frac{\partial \Phi}{\partial n}d\Gamma \qquad (13)$$

In the axisymmetric case, the cross sectional area to be analyzed is divided into triangular ring elements. The test function is expressed as the same form as equation (10) with three nodes. The final discretized form is the same as equation (11).

4.2 Formulation in transformed coordinates

In the transformed coordinates, the Helmholtz equation is obtained from equation (5). For the cylindrical coordinates, the expression is

$$\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{\Phi}}{\partial \tilde{r}}\right)+\frac{\partial}{\partial \tilde{z}}\left(\tilde{r}\frac{\partial \tilde{\Phi}}{\partial \tilde{z}}\right)+\tilde{r}\tilde{k}^2\tilde{\Phi}=0 \qquad (14)$$

The normal derivative to the boundary or flux can be expressed as

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{n}}=\left(\frac{\partial \Phi}{\partial n}\frac{\partial n}{\partial \tilde{n}}-j\tilde{k}M_z\Phi\frac{\partial \tilde{z}}{\partial \tilde{n}}\right)e^{-j\tilde{k}M_z\tilde{z}} \qquad (15)$$

So the boundary conditions corresponding to equation (6) are

$$\begin{cases} \frac{\partial \tilde{\Phi}}{\partial \tilde{n}} = U_0\sqrt{1-M_z^2}-j\tilde{k}M_z\tilde{\Phi} & (z=0) \\ \frac{\partial \tilde{\Phi}}{\partial \tilde{n}} = -j\tilde{k}\frac{1+\beta_T M_z}{\beta_T+M_z}\tilde{\Phi} & (z=\ell) \\ \frac{\partial \tilde{\Phi}}{\partial \tilde{n}} = 0 & \text{(wall boundary)} \end{cases} \qquad (16)$$

Galerkin's procedure for equation (14) leads to the expression of weak form of

$$2\pi\iint\left(\tilde{r}\frac{\partial \tilde{\varphi}}{\partial \tilde{r}}\frac{\partial \tilde{\Phi}}{\partial \tilde{r}}+\tilde{r}\frac{\partial \tilde{\varphi}}{\partial \tilde{z}}\frac{\partial \tilde{\Phi}}{\partial \tilde{z}}-\tilde{r}\tilde{k}^2\tilde{\varphi}\tilde{\Phi}\right)d\tilde{r}d\tilde{z} = 2\pi\int\tilde{r}\tilde{\varphi}\frac{\partial \tilde{\Phi}}{\partial \tilde{n}}d\Gamma \qquad (17)$$

where $\tilde{\varphi}$ is a weighting function. The test function is chosen as

$$\tilde{\Phi}=\{N\}^T\{\tilde{\Phi}\}_e \qquad (18)$$

where $\{\tilde{\Phi}\}_e$ is the nodal velocity potential vector in the transformed coordinates. The discretized equation in the transformed coordinates is derived as

$$\left([\widetilde{M}]_e-\tilde{k}^2[\widetilde{K}]_e+j\tilde{k}[\widetilde{J}]_e\right)\{\tilde{\Phi}\}_e = U_0\sqrt{1-M_z^2}\{\widetilde{W}\}_e \qquad (19)$$

where $[\widetilde{M}]_e, [\widetilde{K}]_e$ and $[\widetilde{J}]_e$ are inertance, elastance and damping matrix associated with the termination wall in the transformed coordinates and $\{\widetilde{W}\}_e$ is the distribution vector in the transformed coordinates. The discretized equation is the same as the ordinary finite element expression. This means that the finite element program developed for the standard Helmholtz equation can be used without modification, but with the potential or the boundary conditions re-defined as given in equations (4) and (16).

5 BOUNDARY ELEMENT FORMULATION

The formulation here presented is somewhat modified from that of Zhenlin [5]. To avoid confusion, the Helmholtz equation in the transformed coordinates (equation (5)) is used instead of equation (2). The boundary element integral expression corresponding to equation (5) is

$$C_i\tilde{\Phi}_i=\int_\Gamma\left(\tilde{\Phi}^*\frac{\partial \tilde{\Phi}}{\partial \tilde{n}}+\frac{\partial \tilde{\Phi}^*}{\partial \tilde{n}}\tilde{\Phi}\right)d\Gamma \qquad (20)$$

where $\tilde{\Phi}^*$ is the fundamental solution

$$\tilde{\Phi}^*=\frac{1}{4\pi\tilde{R}}e^{-j\tilde{k}\tilde{R}} \qquad (21)$$

which is Green's function without particular boundary condition imposed and $\tilde{R}$ is the distance from a source point to the consideration point. C_i is a coefficient related to the solid angle at point i, which becomes 1/2 if the boundary is smooth. By dividing the boundary Γ into the surface boundary elements to execute an integral evaluation of equation (20), the discretized equation is obtained for the velocity potential $\tilde{\Phi}$ and the flux $\tilde{q}$ defined at the element nodes [9], [10], so that

$$[\widetilde{H}]\{\widetilde{\Phi}\} = [\widetilde{G}]\{\widetilde{q}\} \tag{22}$$

By the help of equations (4) and (15), the equation (22) can be transformed into the expression at original coordinates,

$$[H]\{\Phi\} = [G]\{q\} \tag{23}$$

where the components of $[H]$ and $[G]$ are

$$H_{ij} = \widetilde{H}_{ij}e^{-j\widetilde{k}M_z\widetilde{z}_j} + j\widetilde{k}M_z\frac{\partial\widetilde{z}}{\partial\widetilde{n}}\widetilde{G}_{ij}e^{-j\widetilde{k}M_z\widetilde{z}_j} \tag{24}$$

$$G_{ij} = \widetilde{G}_{ij}e^{-j\widetilde{k}M_z\widetilde{z}_j}\frac{\partial n}{\partial\widetilde{n}} \tag{25}$$

The equation (23) shows that the special care is not required for the boundary condition at the expense that the re-evaluation of the components in the coefficient matrices are made. The boundary element program developed for 3-D acoustic field [9] cannot be used as it is, which must be modified so that the coefficient matrices are re-evaluated.

6 NUMERICAL DEMONSTRATIONS

For the numerical demonstrations we consider two ducts: one is a simple duct as shown in Fig.2 in which three models are given, and another is a muffler with expansion-chamber as shown in Fig.3 in which two models are given. The length of the ducts is $\ell = 120$mm and the radius is 10mm but the expansion chamber whose radius is 20mm. The duct is driven at one end ($z = 0$) by the uniform velocity $U_0 = 1$m/sec and at another end ($z = \ell$) it is terminated by the sound absorber with the surface acoustic impedance Z_T. Other wall boundary is assumed to be rigid.

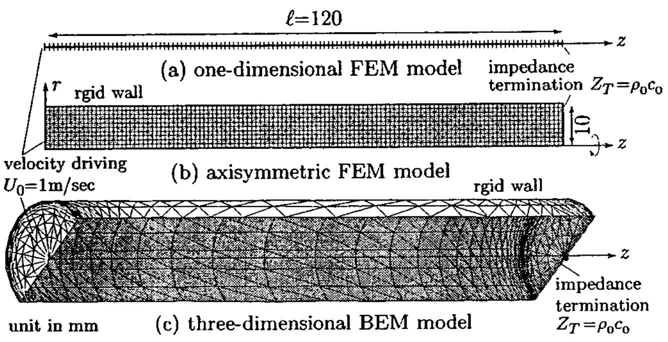

Fig. 2 A simple duct

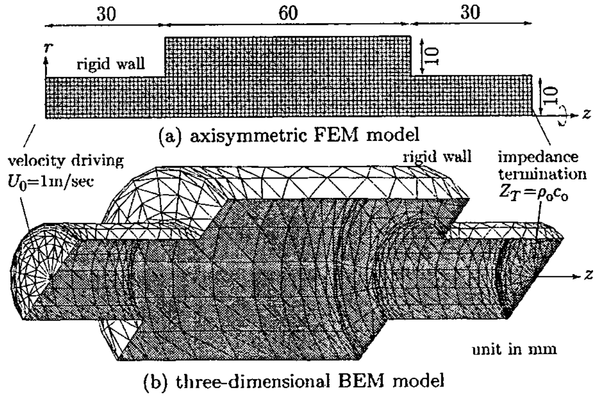

Fig. 3 Muffler

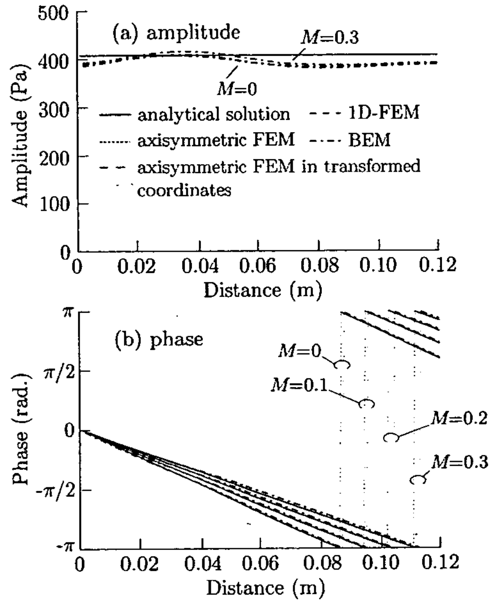

Fig. 4 Sound pressure and phase along the central z axis in a simple duct.

6.1 A simple duct

The sound pressure distributions along the central z axis are shown in Fig.4 when the duct is terminated by the characteristic impedance $Z_T = \rho_0 c_0$. No reflection from the termination is occurred because the end is terminated by the characteristic impedance. The effective wavelength becomes longer as the mach number increases. The finite element solutions well agree with the analytical solution (8) within the error of 0.03%. In the boundary element solution, the slight reflection is observed due to the improper edge discretization. The error is smaller in phase than the amplitude. The boundary element solution for the wavelength is evaluated slightly longer than the analytical solution. The

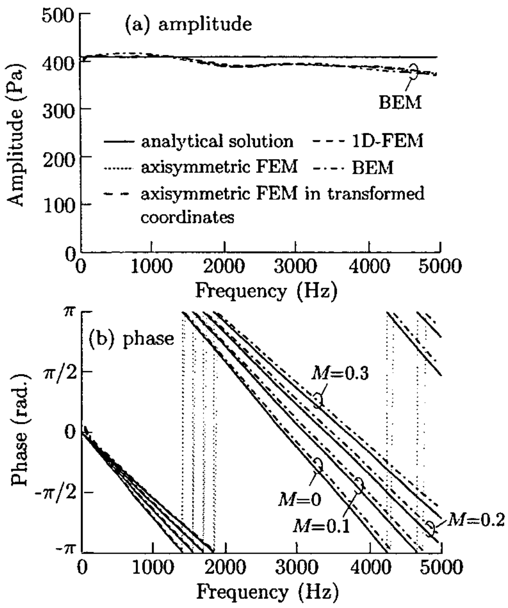

Fig. 5 Frequency characteristics of the sound pressure at the terminated end in the simple duct.

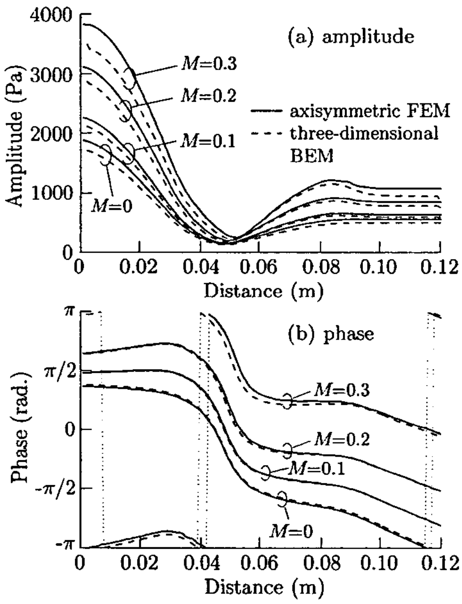

Fig. 6 Sound pressure and phase along the central z axis in the muffler.

transfer frequency characteristics for the sound pressure at the center of the termination wall are shown in Fig.5.

6.2 A muffler with expansion chamber

Fig.6 shows the sound pressure distributions of the muffler when mean flow is uniformly present. There are some differences in amplitude between finite element and boundary element solutions but

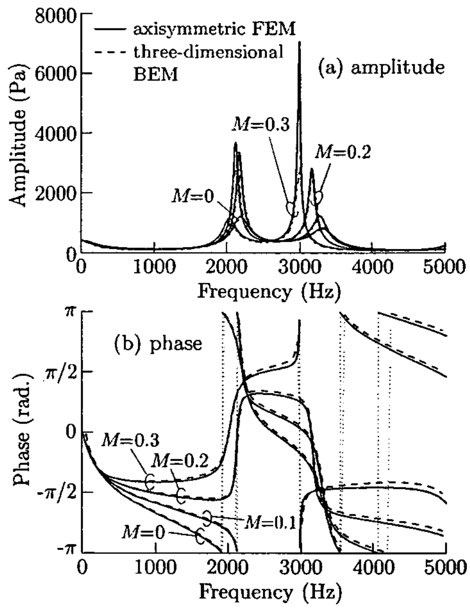

Fig. 7 Frequency characteristics of the sound pressure at the terminated end in the muffler.

the differences are small in phase. Fig.7 shows the frequency transmission characteristics of the muffler when mean flow is uniformly present. The resonance peaks move lower in frequency as the Mach number increases.

6.3 Characteristics when the partial mean flow is present

In conventional muffler systems, the flow is restricted only in the central region and may not present in the expanded part of the chamber. The partial mean flow can easily be incorporated in the modelling with FEM. In order to consider the partial mean flow in the boundary element modelling, the field is divided into two domains as shown in Fig.8 in which the simple duct with the uniform mean flow (domain Ω_1) is coupled to the expansion

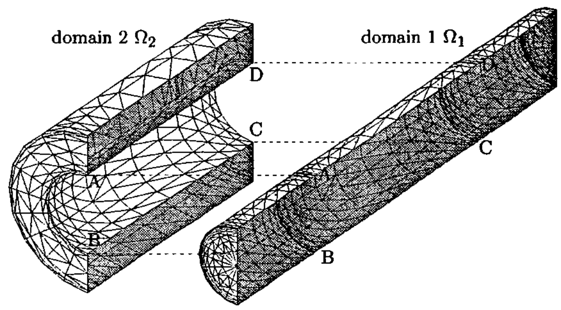

Fig. 8 Partitioned boundary element model of a muffler.

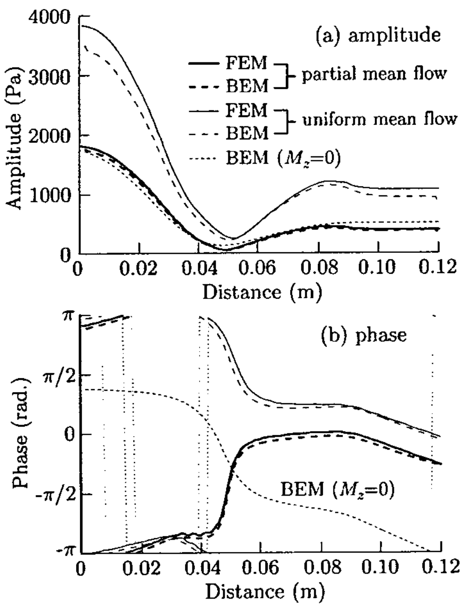

Fig. 9 Comparison between the sound pressure distributions with the uniform mean flow and the partial mean flow for $M = 0.3$.

chamber without flow (domain Ω_2). Continuity conditions are imposed over the connecting surface.

Fig.9 shows the sound pressure distributions both with the uniform mean flow and the partial mean flow for $M = 0.3$. Reasonable agreement is achieved. When the mean flow is partially present, the amplitude decreases and comes closer to the one with no flow.

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BIOGRAPHIES

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