A simulation/analytic approach for queueing network analysis

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Abstract

In this study, we try to improve the accuracy of QN-GPH by the help of simulation approach. We first establish an estimation method for GPH distributions with sufficient accuracy based on empirical distribution, and then perform a brief trial run to find appropriate empirical distributions. After getting GPH form of distributions, we continue the QN-GPH analytic steps and compute necessary performance measures. We apply the method to find sojourn time distributions in a 8-node queueing system and compare the results with the whole simulation and the original two-parametric approximation.

1. Introduction

Queueing networks are very useful modeling tool for the performance analysis of various manufacturing and telecommunication systems. But it is not an easy task to get various performance measures accurately even in a moderate sized queueing network. QN-GPH(Queueing network analysis tool base on GPH method) is an analytic method useful for the delay analysis of queueing networks under the quite general setting. It does not assume Poisson arrivals and exponential service times at each node and gives not only means and variances but also probability distributions. But still, it has its limitation because it uses a two-parameter fitting method for GPH distributions.

QN-GPH method consists of the following procedures.

1. Estimate the traffic parameters(or interarrival time distributions) of external arrivals at each node.
2. Find the GPH approximation for the total arrival processes at each node.
3. Find the GPH approximation for the service time distribution at each node.
4. Perform the decomposition analysis for the performance measures(mean and variance) at each node.
5. Compute the GPH form of the sojourn time distribution of GPH/GPH/1 queue at each node.
6. Compute the total sojourn time distribution in the network for a typical customer. In this step, the distribution of the time to go through some designated route can also be computed.

In this study, we try to improve the accuracy of QN-GPH by the help of simulation approach. That is, we will skip the step 1-4 and instead, apply simulation to estimate corresponding distributions. To accomplish this, we first establish an estimation method for GPH distributions with sufficient accuracy based on empirical distribution, and then perform a brief trial run to find appropriate empirical distributions. After getting GPH form of distributions, we continue the QN-GPH analytic step 5 and 6 and compute necessary performance measures. We will apply the method to find delay time distributions in a queueing system with 8 node obtained from a mobile switching system and compare the results with the whole simulation and the original two-parametric approximation.

2. QN-GPH and GPH fitting

2.1. GPH Semi-Markov chain modeling

Consider an open queueing network. After setting the arrival time epoch of a typical customer as 0, let Z(t) denote the node number at which the customer resides at time t. The state space of this process will be {0, 1, ..., N}, where state 0 implies the outside of the network. The total sojourn time of the customer, which implies the total time spent by the customer in the network, can be represented by

$$T=\inf\{t>0, Z(t)=0\}. \tag{1}$$

Letting N_i denote the number of visits to node i of the customer and T_{ir} denote is the customer's sojourn time at node i at his rth visit, we can rewrite (1) as

$$T=\sum_{i=1}^{N}\sum_{r=1}^{N_i} T_{ir} \ . \tag{2}$$

It is too hard to derive the distribution of T under the general setting. If we assume for fixed node i, $T_{ir}, r=1,2,\ldots$ are iid and for $i\neq j$, T_{ir} and T_{jr} are mutually independent, {Z(t), t>0} forms a semi-Markov chain. Futher if we treat each node as a separate GPH/GPH/1 queueing system, then T_{ir} becomes a waiting time in the GPH/GPH/1 queue corresponding to the ith node, which can be proven to be another GPH random variable(Shanthikumar, 1985). Following the procedures described in Yoon (1994), we can compute the approximate distribution of T_{ir}, which is the sojourn time at node i, at least approximately. Now, the distribution of T which is total sojourn time in the network can be obtained directly from (1) instead of (2). This can be done by deriving distribution of the first passage time to the absorbing state 0(outside) in the corresponding semi-Markov chain. The specific method is described in Yoon(1998) and constitute the essence of QN-GPH.

2.2. GPH fitting

Let $E_0 = 0$ w.p.1 and E_n, n=0, 1, 2, ... be iid exponential rv's with rate λ, and L be a discrete random variable with probability mass function $g = (g(n), n = 0, 1, 2, \ldots)$. Define $X = \sum_{n=0}^{L} E_n$. Then, its distribution function F(x)=P{ $X \le x$} will be

$$F(x) = \sum_{n=0}^{\infty} G(n) \frac{e^{-\lambda x}(\lambda x)^n}{n!}, \quad x \ge 0, \tag{3}$$

where $G(n) = \sum_{i=0}^{n} g(i)$,n=0,1,2,... is the cumulative probability distribution of L. In Shanthikumar (1985), the distribution $F(x)$ is called a *generalized phase-type(GPH) distribution.*

Now defining

$$F_\lambda(t) = \sum_{n=0}^{\infty} F\left(\frac{n}{\lambda}\right) \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad t \ge 0, \ \lambda > 0, \tag{4}$$

we can observe(Yoon 1998) that

$$F_\lambda(t) \to F(t) \quad as \quad \lambda \to \infty. \tag{5}$$

Suppose that we have a sample of size m, say $U_i,\ i = 1, 2, \ldots, m$ from an unknown distribution $F(x)$, $x \ge 0$. The natural estimator of $F(x)$ is its empirical distribution defined by

$$F_m(x) = \frac{\sum_{i=1}^{m} I(U_i \le x)}{m} \tag{6}$$

where $I(A) = 1$ if A is true and $= 0$ if not. Applying the strong law of large numbers, it is easily seen that

$$F_m(x) \to F(x) \quad w.p.1 \quad as \quad m \to \infty \tag{7}$$

Now we may substitute $F_m(x)$ for $F(x)$ into (4) to get

$$F_{m,\lambda}(x) = \sum_{n=0}^{\infty} F_m\left(\frac{n}{\lambda}\right) \frac{e^{-\lambda x}(\lambda x)^n}{n!} \tag{8}$$

which is a GPH. We can easily see

$$F_{m,\lambda}(x) \to F_m(x) \quad as \quad \lambda \to \infty. \tag{9}$$

(9) together with (7) implies any distribution F with nonnegative support can be represented by a GPH and this fact allows us to take $F_{m,\lambda}(x)$ as an approximation for $F(x)$. In the empirical study of Yoon et al.(1994), it is experimentally shown that with relatively small λ(say, 50), the approximation is very accurate when there are sufficiently large number of data.

3. Simulation/Analytic approach

Originally, QN-GPH follows two moment fitting method to estimate service time distribution and interarrival time distribution at each node because of the complexity of the queueing network dynamics. But, this can be attributed to the main source of the error involved in QN-GPH. Since, as we can see in section 2.2., GPH can be estimated directly from the empirical distribution, we actually do not need to stick to the inaccurate two parameter estimation, only if the experimental data can be available. And the data can be generated by simulation.

At the moment our simulation is direct and crude. We just simulate the given queueing network for long enough using Arena and record the every arrival times at each node. Using the crude data, we get the empirical distribution and then a proper GPH distribution of interarrival times at each node. Then, we apply the QN-GPH to get the total sojourn time distribution. We assume that the external arrival processes and service time distributions at each node are already given.

4. An 8-node queueing network

As an example we use an open queueing network obtained from a mobile switching system. The eternal arrivals occur only at node 1 and 8, whose processes are assumed to be Poisson processes with rate 0.1175 and 0.0608. The transition probability matrix is given in Table 1. The service times are assumed to be distributed as an 2-Erlang distributions with parameter given in Table 2.

In Table 3 and 4 we can see the total sojourn time distributions for the customer arrived at node 1 and 8 initially computed by QN-GPH. The comparison of the results with the full simulation and simulation/analytic approach will be reported soon.

Table 1. transition probabilities

	1	2	3	4	5	6	7	8	0
1	0.401	0.017	0.017	0.087	0.152			0.035	0.291
2			0.5						0.5
3	0.5	0.5							
4	0.5							0.125	0.375
5	0.168				0.46	0.066	0.306		
6					1				
7					1				
8	0.071			0.071				0.536	0.322
0	0.659							0.341	

Table 2. traffic parameters

node	external arrival rate	total arrival rate	mean service time	load factor
1	0.1175	0.3473	1.32	0.458
2	0	0.0118	1.08	0.127
3	0	0.0118	1.08	0.127
4	0	0.0413	0.882	0.364
5	0	0.3143	3.0	0.943
6	0	0.0207	1.0	0.207
7	0	0.0961	3.0	0.288
8	0.0608	0.1572	1.32	0.208

Table 3. sojourn time distribution for arrival at node 8

t(msec)	Pr(T<t)	t(msec)	Pr(T<t)
10	0.5063	190	0.8825
20	0.6869	200	0.8883
30	0.7277	210	0.8939
40	0.7423	220	0.8992
50	0.7582	230	0.9042
60	0.7705	240	0.9090
70	0.7821	250	0.9135
80	0.7931	260	0.9178
90	0.8035	270	0.9218
100	0.8134	280	0.9257
110	0.8227	290	0.9294
120	0.8316	300	0.9328
130	0.8401	310	0.9361
140	0.8481	320	0.9393
150	0.8557	330	0.9422
160	0.8629	340	0.9451
170	0.8698	350	0.9477
180	0.8763	360	0.9503

Table 4. sojourn time distribution for arrival at node 1

t(msec)	Pr(T<t)	t(msec)	Pr(T<t)
1.0	0.0136	11.0	0.7858
2.0	0.0625	12.0	0.8271
3.0	0.1415	13.0	0.8611
4.0	0.2373	14.0	0.8888
5.0	0.3383	15.0	0.9112
6.0	0.4362	16.0	0.9294
7.0	0.5266	17.0	0.9439
8.0	0.6069	18.0	0.9555
9.0	0.6766	19.0	0.9648
10.0	0.7360	20.0	0.9722

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