Behavior of Multiple-Attractor Cellular Automata

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ABSTRACT

We present a detailed analysis of the behavior of a multiple-attractor cellular automata.

I. Introduction

An analysis of the state-transition behavior of group cellular automata(briefly CA) was studied by many researchers ([1], [5], [7], [11]). The characteristic matrix of group CA is nonsingular. But the characteristic matrix of nongroup CA is singular. Although the study of nonsingular linear machines has received considerable attention from researchers, the study of the class of machines with singular characteristic matrix has not received due attention. However some properties of nonsingular CA have been employed in several applications ([6], [8], [10], [11]). In this paper, we present a detailed analysis of the behavior of complemented CA derived from a linear CA by replacing the XORs with XNORs at some (or all) of the cells. Also, we give the specific features displayed in the state-transition behavior of the complemented MACA C' resulting from inversion of next-state logic of some (or all) of the cells of multiple-attractor CA(briefly MACA)C. We call C' the CA corresponding to C. Especially we investigate the behavior of the complemented MACA which the complement vector F is taken in the 0-tree as a nonzero state of a linear MACA. And given 0-basic path of 0-tree and a nonzero attractor α of a MACA C with two predecessor we construct an α-tree of C.

II. Nongroup CA

Definition 2.1 [3]. A state with a self-loop in the state-transition diagram of a nongroup CA are referred to as an attractor.

Remark 2.2. The cycles with length $l(\geq 2)$ in the state-transition diagram of nongroup CA are not attractors.

Lemma 2.3 [4]. In an n-cell noncomplementd CA with characteristic matrix T, the number of attractors is 2^{n-r}, where r is the rank of the $(T \oplus I)$ matrix.

Definition 2.4 [3]. The nongroup CA for which the state-transition diagram consists of a set of disjoint components forming(inverted) tree-like structures rooted at attractors are referred to as multiple-attractor CA(MACA).

Remark 2.5. In case the number of attractors is one we call MACA single-attractor CA(SACA).

The tree rooted at a cyclic state α is called the α-tree

Definition 2.6 [3]. The depth of a CA is defined to be the minimum number of clock cycles required to reach the cyclic state from any nonreachable state in the state-transition diagram of the CA.

Lemma 2.7 [6]. The state 0 in a linear nongroup CA is an attractor.

Since the 0-tree and another tree rooted at a nonzero cyclic state have very interesting relationships, the study of the 0-tree is necessary and very important.

Lemma 2.8 [6]. If d is the dimension of the null space N(T) of the characteristic matrix T of a nongroup CA, the total number of 1-predecessors of the state 0 is 2^d

Theorem 2.9 [6]. The number of predecessors of a reachable state and the number of predecessors of the state 0 in a linear nongroup CA are equal.

Definition 2.10 [4]. A state X at level $l(l \leq depth)$ of the α-tree is a state lying on that tree it evolves to the state α exactly after l-cycles (l is the smallest possible integer for which $T^l X = \alpha$)

Definition 2.11 [4]. A state Y of an n-cell CA is an r-predecessor($1 \leq r \leq 2^n - 1$) of a state X if $T^r Y = X$, where T is the characteristic matrix of the CA.

III. Behavior of complemented MACA derived from a linear MACA

In this section we present the behavior of complemented MACA derived fro a linear MACA. Especially we investigate the behavior of complemented MACA form a MACA C which the complement vector F is taken in the 0-tree as a nonzero state of C.

Lemma 3.1 [10]. Let $\overline{T}^p$ denote p times application of the complemented CA operator $\overline{T}$. Then,

$$\overline{T}^p f(x) = [I \oplus T \oplus T^2 \oplus \cdots \oplus T^{p-1}] [F(x)] \oplus [T^p][F(x)]$$

where T is the characteristic matrix of the corresponding noncomplemented rule vector and [F(x)] is an n-dimensional vector (n=number of cells) responsible for inversion after XNORing. F(x) has '1' entries (i.e. nonzero entries) for CA cell positions where XNOR function is employed.

Lemma 3.2. Let $\overline{T}^p$ denote p times application of the complemented CA operator $\overline{T}$. Then,

$$\overline{T}^p[F(x)] = [T^p \oplus T^{p-1} \oplus \cdots \oplus T^2 \oplus T \oplus I][F(x)]$$

Lemma 3.3. Let C be a linear MACA with depth d and F be a state at the level i ($0 < i \leq d$) of the 0-tree in C as a complemented vector. Then $\overline{T}^{i-1}F$ is an attractor in the complemented MACA C′ corresponding to C.

Lemma 3.4. Suppose that there exists at least one attractor in the complemented CA C′ corresponding to an n-cell linear MACA C with k attractors. Then the number of attractors in the complemented CA C′ is the same as that in the original linear one.

Remark 3.5. From Lemma 3.4. we know that if C is a linear MACA, then the complemented CA corresponding to C is also a MACA.

Theorem 3.6. In the state-transition diagram of the complemented MACA C′ corresponding to a linear MACA C, the sum of different predecessors of any reachable state is the nonzero 1-predecessor of the state 0 of C.

Theorem 3.7. Let dim (N(T))=1. Let C be a linear MACA. Let F be a state at level $l(>0)$ in the 0-tree of C and C′ be the corresponding complemented MACA. Then the states of 0-tree of C are rearranged in the state-transition diagram of C′ as the following:

(a) All states at levels higher than l of C will remain unaltered.

(b) The states at level l of C get rearranged at level up to $(l-1)$ of C′.

(c) The states at levels up to $(l-1)$ of C get rearranged at level l of C′.

(d) F lies on level $(l-1)$ of C′.

Corollary 3.8. Let C be a linear MACA and T the characteristic matrix of C and let F be a nonreachable state in the 0-tree of C. Then the state 0 is a nonreachable state in the state-transition diagram of $\overline{T}$.

Rmark 3.9. In Theorem 3.7 we investigated the rearrangement of the states of 0-tree in the state-transition diagram of C′. By similar methods we can prove the states of another nonzero tree of C are rearranged in the state-transition diagram of C′ as the rearrangement of the states of 0-tree.

Ⅳ. Construction of a tree from a given 0-tree in a linear MACA

In this section we construct an α-tree of a MACA C with two predecessor if we knew a 0-basic path of the 0-tree and a nonzero attractor α of C.

Theorem 4.1 [2]. Let C be a linear SACA having two-predecessor. If the states of the state-transition diagram of C are labeled such that $S_{l,k}$ be the $(k+1)$-th state in the l-th level, then the following hold:

$$S_{l,k} = S_{l,0} \oplus \sum_{i=1}^{l-1} b_i \, S_{i,0}$$

where $b_{l-1}\, b_{l-2} \cdots b_1$ is the binary representation of k and the maximum value of k is $2^{l-1}-1$.

Definition 4.2. Let C be a linear MACA with two-predecessor and the depth of C be d.

Let β be a nonreachable state of the α-tree of C.

Remark 4.3. Let C be the linear SACA in Theorem 4.1 with the depth d. Then

$S_{d,0} \to S_{d-1,0} \to \cdots \to S_{1,0} \to 0$

is a 0-basic path of the 0-tree of C.

Lemma 4.4. Let C be a linear MACA with two-predecessor. Let $\alpha_{i,j}$ (*resp.* $\beta_{i,j}$) be the $(j+1)$-th state in the i-th level of the α-tree (resp. β-tree) in C. Then

$$\alpha_{i,j} \oplus \beta_{i,j} = \alpha \oplus \beta$$

Corollary 4.5. Let C be a linear MACA with two-predecessor (depth=d) and T be the characteristic matrix of C.

If $S_{d,0} \to S_{d-1,0} \to \cdots \to S_{1,0} \to 0$ is a 0-basic path of the 0-tree of C, then

$$(S_{d,0} \oplus \alpha) \to (S_{d-1,0} \oplus \alpha) \to \cdots \to (S_{1,0} \oplus \alpha) \to \alpha$$

is a α-basic path of the α-tree of C.

Example 4.6. Let C be a five-cell linear MACA with the rule <204, 240, 240, 240, 240>. Then

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The minimal polynomial m(x) of T is m(x)= $x^4(x+1)$ and attractors are 0 and 31.

The state-transition diagram is as the following:

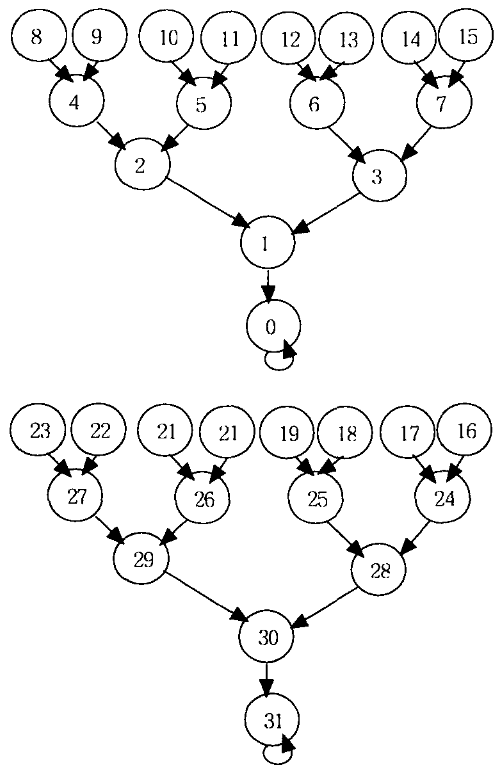

Figure 1: The state-transition diagram of C

8→4→2→1→0 is a 0-basic path. The 31-basic path corresponding to the 0-basic path is 23→27→29→30→31.

The following theorem is an extension of Theorem 4.1.

Theorem 4.7. Let C be a linear MACA having two-predecessor. If the states of the state-transition diagram of C are labeled such that $S^{\alpha}_{l,k}$ ($resp.$ $S_{i,k}$) be the $(k+1)$-th state in the l-th level of the α-tree (resp. 0-tree) in C, then the following hold:

$$S^{\alpha}_{l,k} = S^{\alpha}_{l,0} \oplus \sum_{i=1}^{l-1} b_i S_{i,0}$$

where $b_{l-1} b_{l-2} \cdots b_1$ is the binary representation of k and the maximum value of k is $2^{l-1}-1$.

Theorem 4.8. Let C be a linear MACA with two-predecessor (depth=d) and F be a level i state $(1 \leq i \leq d)$ in the 0-tree of C. Let x be a nonreachable state of C such that $T^{d-i}x = F$. Then a $\overline{T}^{i-1}x = F$-basic path of the $\overline{T}^{i-1}x = F$-tree of C′ is

$$x \rightarrow Tx \rightarrow \ldots \rightarrow T^{d-(i+1)}x \rightarrow 0 \rightarrow F \rightarrow \overline{T}F \rightarrow \cdots \rightarrow \overline{T}^{i-1}F$$

Corollary 4.9. Let C be a linear SCAC with the depth d and F be a nonreachable state in the 0-tree of C. Then $0 \rightarrow F \rightarrow \ldots \rightarrow \overline{T}^{d-1}F$ is a $\overline{T}^{d-1}F$-basic path of C′.

V. Conclusion

We give the specific features displayed in the state-transition behavior of the complemented MACA C′ resulting from inversion of next-state logic of some (or all) of the cells of MACA C. Especially we investigate the behavior of the complemented MACA which the complement vector F is taken in the 0-tree as a nonzero state of a linear MACA. And given 0-basic path of 0-tree and a nonzero attractor α of a MACA C with two predecessor we construct an α-tree of C.

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