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Analytical Solution for Flow Field by Arbitrarily-Located Multi Injection-Pumping Wells

In Wook Yeo · Kang-Kun Lee
School of Earth and Environmental Sciences, Seoul National University
iwyeo@snu.ac.kr · kklee@snu.ac.kr

Abstract

Analytical solutions have been derived to delineate the capture zone created by pumping wells for the remediation design of contaminated groundwater. These previous analytical solutions are often restricted to pumping wells only, specific well locations, a limited number of wells, and an isotropic aquifer. Analytical solution was developed to deal with arbitrarily located multi injection-pumping wells in an anisotropic homogeneous aquifer. The solution presented in this study provides a simple, easy method for determining the complex flow field caused by multi injection-pumping wells at different rates, and will consequently be useful in pump-and-treat design.

key words: analytical solution, multi-wells, anisotropic aquifer, capture zone

1. Introduction

The pump-and-treat method has been considered an effective technology for the remediation of contaminated groundwater, in that pump-and-treat removes significant quantities of contaminants in a relatively short time, and prevents the contamination of aquifer that are not yet contaminated. Optimal design of contaminated groundwater capture system is a key for the success of pump-and-treat. Analytical solutions have been suggested to determine capture zones and stagnation points [*Javandel and Tsang*, 1986; *Grubb*, 1993; *Shan*, 1999]. However, these analytical solutions are restricted to simple problems such as an isotropic homogeneous aquifer with pumping only, and there is also a limitation to the number of pumping wells and specific well locations. Reinjection of treated groundwater is often necessary to reduce the time required to achieve cleanup levels of contaminants, and to reduce remediation costs by speeding the flushing of contaminants. Multi injection-pumping wells result in a complex flow field that previous analytical solutions cannot handle.

We assume that a confined aquifer is homogeneous but anisotropic, and of constant in thickness, and also assume that groundwater flow is in the steady-state. Based on these assumptions, an analytical solution is developed to calculate the flow field of arbitrarily located multi-wells. The solution presented in this study provides a useful information for the effective pump-and-treat design in a complex flow field with a large number of multi injection-pumping wells at different injection-pumping rates.

2. Analytical Solution

The geometry of Figure 1 is considered to derive an analytical solution. Fluid can be injected and pumped at any arbitrary point. The following equation governs fluid flow with multi injection-pumping wells [*de Marsily*, 1986]:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = \sum_{i=1}^{p} Q_i \delta(x - x_i, y - y_i) \tag{1}$$

where T_x and T_y are the hydraulic transmissivities in the x and y directions, h is hydraulic head, x_i and y_i are the coordinates of the injection or pumping wells, p is the number of wells, δ is the Dirac delta function, and Q_i is the injection or pumping rate of the ith well, defined to be negative for injection and positive for pumping.

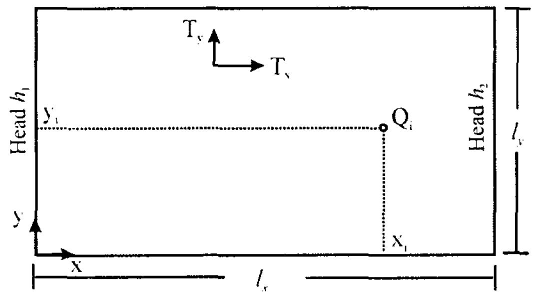

Figure 1. Aquifer and flow geometries assumed for the derivation of analytical solution. Injection and pumping wells can be arbitrarily located.

Hydraulic heads at $x = 0$ and $x = l_x$ are kept constant at h_1 and h_2, respectively. The head distribution can be expressed as

$$h(x,y) = h'(x,y) + \frac{(h_2 - h_1)}{l_x} x + h_1 \tag{2}$$

where $h'(x,y)$ accounts for head change caused by the injection-pumping wells, and the other terms on the right side account for the applied heads at the boundaries. The component $h'(x,y)$ satisfies the following boundary conditions:

$$h'(0,y) = h'(l_x, y) = h'(x,0) = h'(x, l_y) = 0 \tag{3}$$

The faces of $y = 0$ and $y = l_y$ is initially assumed to be constant head boundaries, so as to derive a more accommodative analytical solution for both constant head and no-flow boundaries at these faces. These faces can be treated as no-flow boundaries using image well theory, which is addressed below in the part of example.

Substituting equation (2) into equation (1) leads to

$$T_x \frac{\partial^2 h'}{\partial x^2} + T_y \frac{\partial^2 h'}{\partial y^2} = \sum_{i=1}^{p} Q_i \delta(x - x_i, y - y_i) \tag{4}$$

The boundary conditions of equation (3) indicate that $h'(x,y)$ can be represented by sine in x and sine in y over the aquifer. Therefore, the following double Fourier transforms can be defined:

$$F_{ss}(h'(x,y); x \to n, y \to m) = \int_0^{l_x} \int_0^{l_y} h'(x,y) \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} \, dy dx \tag{5}$$

Taking sine transforms in x and y of each term in the differential equation (4) yields

$$\int_0^{l_x}\int_0^{l_y} T_x \frac{\partial^2 h'}{\partial x^2} \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y}\, dydx + \int_0^{l_x}\int_0^{l_y} T_y \frac{\partial^2 h'}{\partial y^2} \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y}\, dydx$$

$$= \int_0^{l_x}\int_0^{l_y} \sum_{i=1}^{p} Q_i \delta(x - x_i, y - y_i) \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y}\, dydx \qquad (6)$$

The double Fourier transforms of the second partial derivatives of $h'(x,y)$ can be evaluated by applying one dimensional rules successively [*Churchill*, 1972] as follows:

$$\int_0^{l_x}\int_0^{l_y} T_x \frac{\partial^2 h'}{\partial x^2} \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y}\, dydx = -\frac{n^2\pi^2}{l_x^2} F_{ss}(n, m) \qquad (7)$$

$$\int_0^{l_x}\int_0^{l_y} T_y \frac{\partial^2 h'}{\partial y^2} \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y}\, dydx = -\frac{m^2\pi^2}{l_y^2} F_{ss}(n, m) \qquad (8)$$

Using the following rule for the Dirac delta function [*O'Neil*, 1995]:

$$\int_0^{\infty} f(t)\delta(t-a)dt = f(a) \qquad (9)$$

the double Fourier sine transforms of the source terms are evaluated as

$$\int_0^{l_x}\int_0^{l_y} \sum_{i=1}^{p} Q_i \delta(x - x_i, y - y_i) \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y}\, dydx = \sum_{i=1}^{p} Q_i \sin\frac{n\pi x_i}{l_x} \sin\frac{m\pi x_i}{l_y} \qquad (10)$$

Replacing (6) with (7), (8) and (10), and rearranging it in terms of *Fss* gives

$$F_{ss}(n, m) = -\frac{\sum_{i=1}^{p} Q_i \sin\frac{n\pi x_i}{l_x} \sin\frac{m\pi x_i}{l_y}}{T_x n^2 \pi^2 / l_x^2 + T_y m^2 \pi^2 / l_y^2} \qquad (11)$$

The inverse double Fourier sine transforms of equation (5) can be found by applying one dimensional cases successively:

$$h'(x, y) = \frac{4}{l_x l_y} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{ss}(m, m) \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y} \qquad (12)$$

The analytical solution can be formulated by replacing (2) with (11) and (12) as follows:

$$h(x, y) = \frac{(h_2 - h_1)}{l_x} x + h_1 - \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sum_{i=1}^{p} Q_i \sin\frac{n\pi x_i}{l_x} \sin\frac{m\pi x_i}{l_y}}{T_x \frac{n^2 l_y}{l_x} + T_y \frac{m^2 l_x}{l_y}} \sin\frac{n\pi x}{l_x} \sin\frac{m\pi y}{l_y} \qquad (13)$$

3. Numerical Example

An example is presented to show how analytical solution calculates a flow field caused by multi-wells. Boundary heads at up- and down-streams are 10 m and 0 m and the sides of y = 0 and y = l_y have no flow. To accommodate no flow at these faces, the field is extend vertically above and below the aquifer having 500m×500m (Figure 2). In theory, image wells extend to infinity, but it is known that pairs of image wells closest to real well yield a acceptable head change because others have a negligible influence on the head change. T_x and T_y is 1×10^{5} m^2/s. Two real pumping wells at the rate of 5×10^{5} m^3/s are placed at (350 m, 700 m) and (350 m, 800 m), and one real injection well with the rate of 2×10^{-5} m^3/s is located at (150 m, 750 m). Image pumping and injection wells has the same rates as real wells. Figure 2 shows that contaminated groundwater can be contained in the central area by the combination of two pumping and one injection wells, and also shows that analytical solution can be used for testing pump-and-treat design.

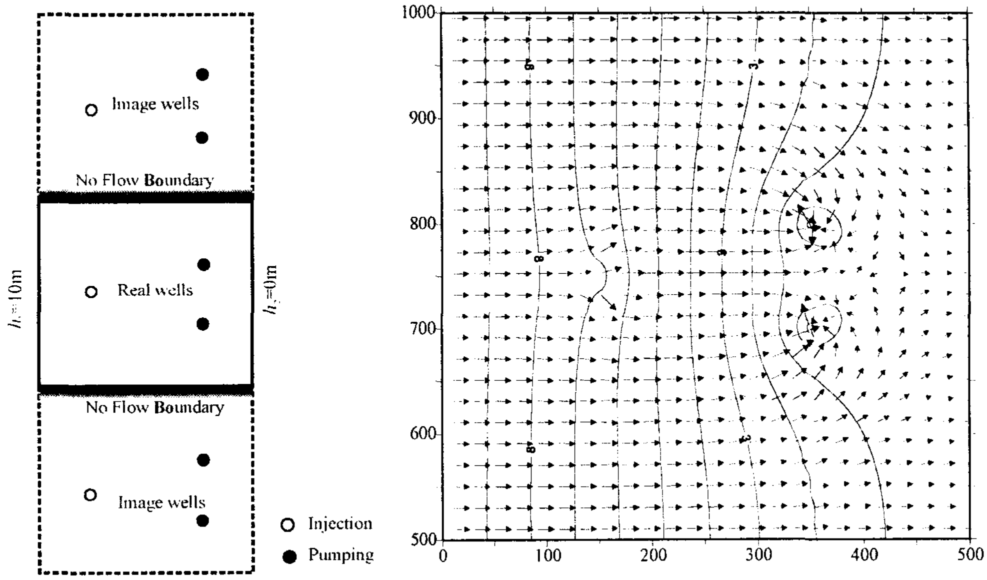

Figure 2. Geometry of real and images wells. Analytical solution calculates head distribution and gradients generated by injection and pumping wells.

4. Conclusion

An analytical solution was presented to deal with multi injection-pumping wells that complicate groundwater flow field, and the analytical solution overcomes some limitations of previous analytical solutions for delineating the capture zone; for examples, numerous wells can be placed at any location, injection is also available, and anisotropic aquifer can be handled. Analytical solution developed in this study is a good alternative to previous analytical solutions especially for a complex flow field by multi injection-pimping wells.

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