Elliptic Integral Solutions of Large Deflection of Reinforcing Fiber Elastica with Circular Wavy Pattern

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Abstract

The solution of two-dimensional deflection of circular wavy reinforcing fiber elastica was obtained for one end clamped boundary under concentrated load condition. The fiber was regarded as a linear elastic material. Wavy shape was described as a combination of half-circular arc smoothly connected each other with constant curvature of all the same magnitude and alternative sign. Also load direction was taken into account. As a result, the solution was expressed in terms of a series of elliptic integrals. These elliptic integrals had two different transformed parameters involved with load value and initial radius of curvature. While we found the exact solutions and expressed them in terms of elliptic integrals, the recursive ignition formulae about the displacement and arc length at each segment of circular section were obtained. Algorithm of determining unknown parameters was established and the profile curve of deflected beam was shown in comparison with initial shape.

Key Words : elliptic integral, Elastica, Wavy Pattern, ignition formulae, initial radius of curvature

Introduction

The solution of large deflection of elastica beam has been frequently studied in various areas for the last century. On the basis of curvature-momentum relation and equilibrium equations, model of the system results in a set of ordinary differential equations. Although the material properties of a beam are supposed to be linear elastic, due to the complicated expression of curvature the model tends to have geometrically non-linear characteristics. In only restricted case - point load, constant initial curvature of beam - one can find an analytical solution such as elliptic integrals.

In the fiber and textile area, if anything, the researches lie in the case of zero initial curvature and small deflection. In applied mathematical area, Conway[1], Nagesawara[2] and many other researchers had taken into account the initial curvature, load direction and complicated boundary conditions. A wavy pattern in fiber is called "crimp". For the materials such as woven, knitted and nonwoven fabrics or their composites, crimped shape of an element is a general characteristics which is to be considered. Thus we found an exact expression of two-dimensional deflection of crimped fiber cantilever with end concentrated load.

Theoretical Modeling

DESCRIPTION OF FIBER CHARACTERISTICS

In this work, fiber is considered as a beam element following such assumptions.

1. Beam is a linear elastic and inextensible material.
2. Beam cross-section is circular and does not change during deformation.
3. Shear stress in the fiber is negligible and fiber cross-section remains perpendicular to deflected line.
4. Wave pattern is coplanar.
5. Wave pattern is a combination of half circular arcs with constant curvature of all the same magnitude and alternative sign.(see fig. 1).

Curvature($_0$) = $\pm\frac{1}{\rho}$ '+' when circular element is upward (1a)

'-' when circular element is downward

Arc length(s_i) = $\frac{\pi\rho}{2}$ (for clamped and loaded segment) (1b)

$\pi\rho$ (for intermediate segments)

P : concentrated load , α : inclined angle

(Fig 1)

GEOMETRICAL PARAMETERS AND FREE BODY DIAGRAM

First, we put the origin at clamped point and construct the rectangular Cartesian coordinates (x-y plane) and put the horizontal distance of any deformed position to x, vertical distance to y,

formula (1c)

(Fig. 2)

MATHEMATICAL SOLUTIONS

The Governing equations of this problem are as follow

formula (2, 3)

And the boundary conditions are

formula (4)

The non-dimensional variables are introduced as follow

formula (5)

The equation (3) can be integrated as

formula (6)

Since the beam is a connected series of the segments with alternatively signed curvature, the integral constant of equation (6) has the different values at each segment. Therefore the procedure goes recursively. The solutions are the forms of recursive "ignition formulae". Numbering example of subscript is shown in fig. 3. Subscript begins from̃ to n. It indicates that the subscript (i-1) is the origin of i-th segment, that is, the i-th segment starts from (i-1)th subscript and ends to i-th one. Clamped boundary is denoted as subscript n. ($1 \le i \le n$)

(Fig. 3)

Using the equation (2) and (4), one can determine integral constant as the following recursive equation.

formula (7)

Where, ω is the slope angle at loaded tip end.

$(\omega = \theta_0$, $X = x_0$ and $Y = y_0)$ and $\Lambda = \frac{P\rho^2}{EI}$

As a result, the solution is obtained for two cases as follow.

Case 1 $(-1 < C_i < 1)$

The arc length information is as follows

formula (8, 9)

Where, $F(k, \phi)$ is the elliptic integral of the first kind

and $\phi_i = \arcsin(\{\frac{1}{2}(\frac{\pi}{2} - \alpha + \theta_i)\}/k_i)$.

The profile informations are as follow

formula (10, 11)

Where, $E(k, \phi)$ is the elliptic integral of the second. If the arbitrary coordinate (x, y) is needed, one can replace the parameters x_i, y_i and ϕ_i with x, y and ϕ.

Case 2 $(C_i < -1)$

formula (12a,b)

The arc length information is as follows

formula (13)

where, $\varphi_i = \frac{1}{2}(\frac{\pi}{2} - \alpha + \theta_i)$.

The profile informations are as follow.

formula (14, 15)

Results and Discussions

This part deals with the calculated results for 1 element strait beam as the justification of our model by comparing the former reproted data, and the results for 2-elemented beam.

One-element straight beam example

The results are shown in Table 1. It shows good agreements with former reported results.[3]

(Table 1)

Two-elements example

The applied load for (a) is 0.5, 1.0, 1.5, 2.0 and 2.5 mPa and 1.0, 2.0, 3.0, 4.0 and 4.5 mPa for (b) respectively. Material properties are E = 40MPa, D = 3mm, R = 10cm α=30° for (a) and E = 100MPa, D = 2mm, R = 5cm, α=45° for (b) respectively

(Fig. 4a, b)

Conclusion

Two-dimensional deflection of crimped fiber elastica problem was modeled and solutions are obtained. We determined the displacement of arbitrary conditions and built the algorithm of calculating end slope ω at given load P. The solutions have the form of recursive ignition formulae. Statically indeterminate characteristics caused the non-linearity of solution.

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Figure Index

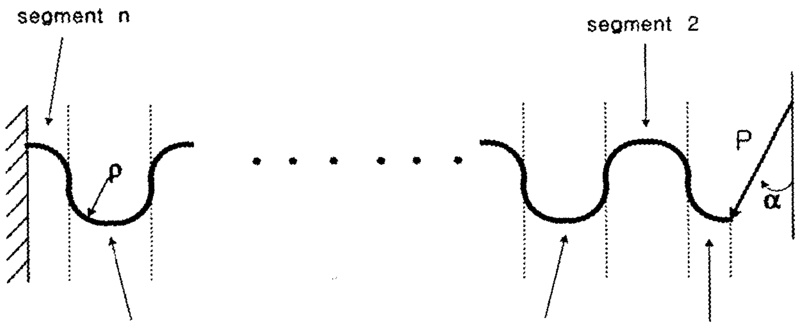

Fig.1 The segment numbering of crimped fiber under load

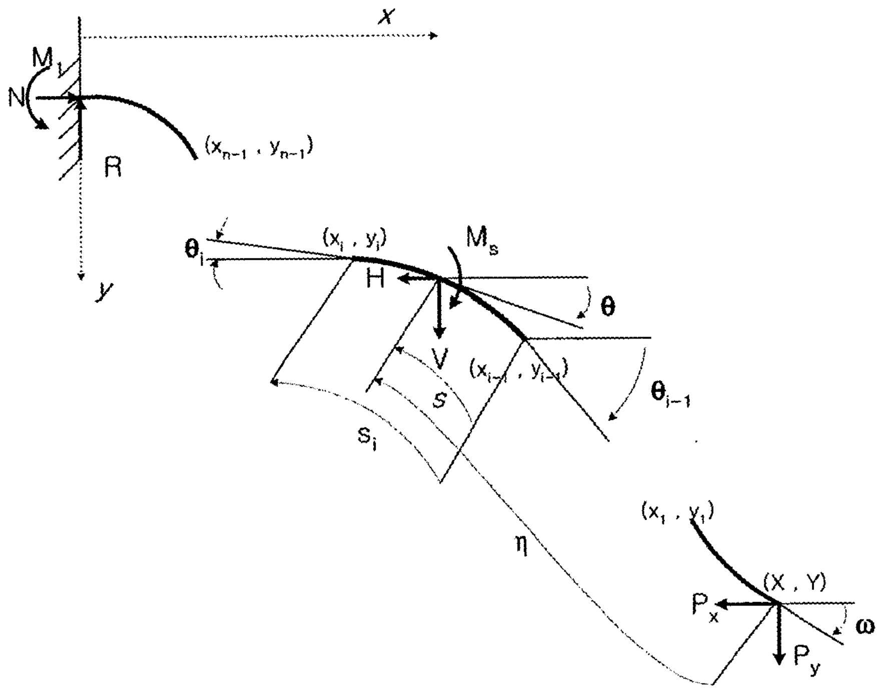

Fig. 2 Free body diagram of fiber element under load

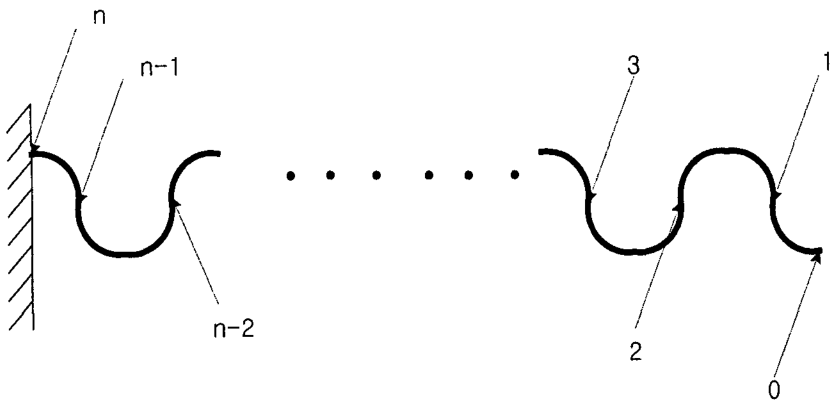

Fig. 3 Numbering configuration of each segment

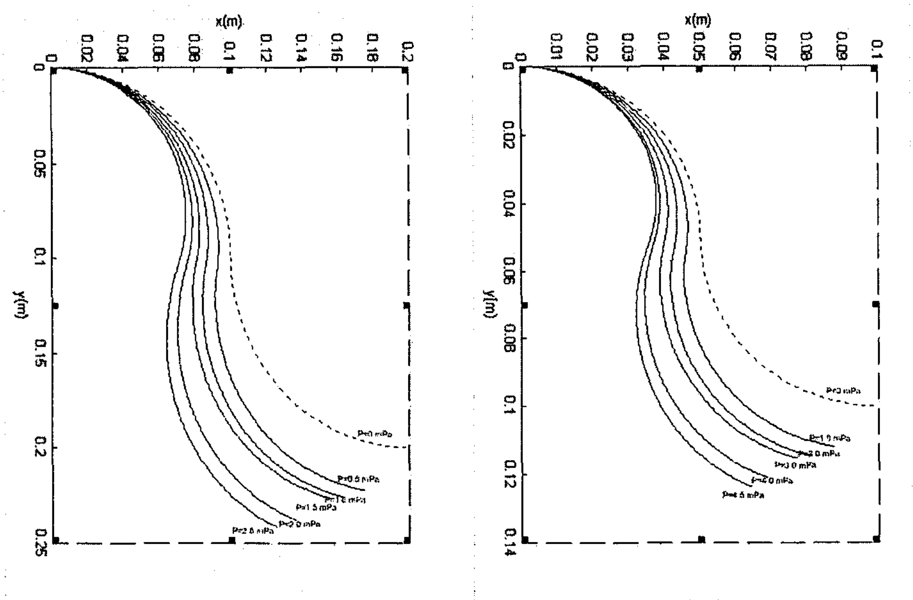

Fig. 4 Solutions of the solutions for (a) E = 40MPa, D = 3mm, R = 10cm α=30° and (b) E = 100MPa, D = 2mm, R = 5cm, α=45°

Formulae Index

$$\frac{dx}{d\eta}=-\cos\theta\ ,\ \frac{dy}{d\eta}=-\sin\theta\ ,\ \theta=\arctan\frac{dy}{dx} \tag{1c}$$

$$\frac{d\theta}{ds} = \pm\frac{1}{\rho} - \frac{1}{EI}\{P_y(X-x)+P_x(Y-y)\} \tag{2}$$

$$\frac{d^2\theta}{ds^2} = -\frac{P}{EI}\cos(\alpha-\theta) \tag{3}$$

$$\theta = 0 \text{ at } x = y = 0, \quad \frac{d\theta}{ds} = \frac{1}{\rho} \text{ at } x = X, y = Y, s = 0 \tag{4}$$

$$\frac{s}{s_i} = \varepsilon_i \ , \ \frac{Ps_i^2}{EI} = \lambda_i \ , \ \bar{X} = \frac{X}{L}, \bar{Y} = \frac{Y}{L}, \bar{x} = \frac{x}{L}, \bar{y} = \frac{y}{L}, \bar{x}_i = \frac{x_i}{L}, \bar{y}_i = \frac{y_i}{L}. \tag{5}$$

$$\frac{1}{2}\left(\frac{d\theta}{d\varepsilon_i}\right)^2 = \lambda_i\{\sin(\alpha-\theta)-C_i\} \tag{6}$$

$$\begin{aligned} C_i &= \sin(\alpha-\theta_{i-1}) - \frac{1}{2}\left[\sqrt{2\{\sin(\alpha-\theta_{i-1})-C_{i-1}\}} - \frac{2}{\sqrt{\Lambda}}\right]^2 \quad (i \ge 2) \\ C_1 &= \sin(\alpha-\omega) - \frac{1}{2\Lambda} \quad (i=1) \end{aligned} \tag{7}$$

$$\sqrt{\lambda_i} = F(k_i, \phi_i) - F(k_i, \phi_{i-1}) \tag{8}$$

$$\sin\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\} = k_i \sin\phi \ , \ k_i = \sqrt{\frac{1-C_i}{2}}. \tag{9}$$

$$\begin{aligned} \frac{\sqrt{\lambda_{i-1}}\cdot L}{s_i}(\bar{x}_{i-1}-\bar{x}_i) &= [2\sin\alpha\{E(k_i, \phi_i)-E(k_i, \phi_{i-1})\} \\ &+ 2k_i\cos\alpha(\cos\phi_{i-1}-\cos\phi_i) - \sin\alpha\{F(k_i, \phi_i)-F(k_i, \phi_{i-1})\}] \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\sqrt{\lambda_{i-1}}\cdot L}{s_i}(\bar{y}_{i-1}-\bar{y}_i) &= [-2\cos\alpha\{E(k_i, \phi_i)-E(k_i, \phi_{i-1})\} \\ &+ 2k_i\sin\alpha(\cos\phi_{i-1}-\cos\phi_i) + \cos\alpha\{F(k_i, \phi_i)-F(k_i, \phi_{i-1})\}] \end{aligned} \tag{11}$$

$$\sqrt{\frac{2}{1-C_i}} = k_i \ , \ \frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right) = \varphi \tag{12a,b}$$

$$(-1)^{i-1}\frac{\sqrt{\lambda_i}}{k_i} = F(k_i, \varphi_i) - F(k_i, \varphi_{i-1}) \tag{13}$$

$$\frac{L}{s_i}(\bar{x}_{i-1}-\bar{x}_i)=(-1)^{i-1}\int_{\varphi_{i-1}}^{\varphi_i}\frac{k_i\cos(2x+\alpha-\frac{\pi}{2})}{\sqrt{\lambda_i(1-k_i^2\sin^2 x)}}dx$$

$$=(-1)^{i-1}\frac{k_i}{\sqrt{\lambda_i}}\ [\cos\alpha(\sqrt{1-k_i^2\sin^2\varphi_{i-1}}-\sqrt{1-k_i^2\sin^2\varphi_i})+\frac{2\sin\alpha}{k_i^2}\{E(k_i\ ,\ \varphi_i)-E(k_i\ ,\ \varphi_{i-1})\}$$

$$+(1-\frac{2}{k_i^2})\sin\alpha\cdot\{F(k_i\ ,\ \varphi_i)-F(k_i\ ,\ \varphi_{i-1})\}]$$

$$\frac{L}{s_i}(\bar{y}_{i-1}-\bar{y}_i)=(-1)^{i-1}\int_{\varphi_{i-1}}^{\varphi_i}\frac{k_i\sin(2x+\alpha-\frac{\pi}{2})}{\sqrt{\lambda_i(1-k_i^2\sin^2 x)}}dx$$

$$=(-1)^{i-1}\frac{k_i}{\sqrt{\lambda_i}}\ [\sin\alpha(\sqrt{1-k_i^2\sin^2\varphi_{i-1}}-\sqrt{1-k_i^2\sin^2\varphi_i})-\frac{2\cos\alpha}{k_i^2}\{E(k_i\ ,\ \varphi_i)-E(k_i\ ,\ \varphi_{i-1})\}$$

$$-(1-\frac{2}{k_i^2})\cos\alpha\cdot\{F(k_i\ ,\ \varphi_i)-F(k_i\ ,\ \varphi_{i-1})\}]$$

(14), (15)

Table Index

λ_1	$\omega/(\frac{\pi}{2})$	$1-\bar{X}$	$\bar{Y}$
0	0	0	0
0.2	0.06342	0.066	0.003
0.4	0.12550	0.131	0.010
0.6	0.18507	0.192	0.022
0.8	0.24130	0.249	0.038
1.0	0.29369	0.302	0.056
1.5	0.40727	0.411	0.108
2	0.49773	0.493	0.161
3	0.62779	0.603	0.254
4	0.71380	0.670	0.329
5	0.77371	0.714	0.388
6	0.81725	0.745	0.435
7	0.84993	0.767	0.473
8	0.87515	0.785	0.505
9	0.89501	0.799	0.532
10	0.91074	0.811	0.555

Table 1. Angle of rotation and deflections for one-element-straight-beam case