Simulation of Compressible Stratified Flow by the Finite Difference Lattice Boltzmann Method

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차분격자볼츠만법을 이용한 압축성성층유체의 수치계산

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중력이 작용하는 압축성유체를 고려함에 있어, 밀도성층 뿐만아니라 엔트로피 성층의 고려도 중요하다. 본 연구에서 압축성격자볼츠만 유체모델을 이용한 차분계산법을 이용하여 2차원 채널에서 성층류의 전형적인 형상인 선택취수현상을 시뮬레이션 하였으며, 본 모델의 유효성을 확인하였다. 또한 비점성, 비압축성유체의 선택취수흐름과의 차이에 관해서 압축성의 관점에서 고찰하였다.

1. Introduction

In recent years, the interest to the lattice Boltzmann method (LBM) has been rapidly increasing, and the number of applications has also been increasing from conventional ordinary fluids to complex fluids.

The lattice Boltzmann method (LBM) is a method to simulate motions of continuous fluids by computing the collision and the propagation of microscopic particles. This method has been developed from the lattice gas automata (LGA)[1-2]. Essentially the technique is considered to simulate the Navier-Stokes equations, and actually the Navier-Stokes equations are derived by the so-called Chapmann-Enskog expansion technique. But the particles translate from one lattice site to neighbor site without collision, and the lattice size can be considered to be the counter part of the order of the mean free path in the molecular gas dynamics. The lattice size is not necessarily small compared with the length scale.

The finite difference lattice Boltzmann method (FDLBM) is one of the computational fluid mechanics methods which is developing from the LBM[3-4]. In the LBM, fluid is regarded as gathering of many particles repeating collision and translation, and the motion of macroscopic fluid is expressed by calculation of those two motions of particles.

Although by using LBM until now, authors examine the flow in which the gravity is important such as the natural convection, the thermo hydrodynamic model is developed and has been verified[5]. However, LBM have many problems such as becoming unstable numerically to a heat flow problem or a high Reynolds number flow.

But, by making the lattice from in dispersion of space, and the physical form of particle movement separate, FDLBM became possible and easy to calculate for complicated object

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form, and the application of its to various flow places was attained.

The effect of the gravity in the compressible fluid is greatly different from that of the incompressible fluid. In case of the incompressible fluid, the fluid is stable when the density increases in downward, and it is unstable when reversed. However, the density changes by the pressure when the fluid is compressible, and in the criterion of above, it is insufficient.

For the compressible fluid, though there are many cases in which concepts such as the potential temperature are used, but it is convenient to introduce the entropy as a general concept used and consider the stratification of the entropy[6].

The criterion of the stability is that the fluid is stable when the entropy increases upward, if the fluid motion changes in iso-entropic manner, and unstable otherwise.

Stratified flows have been studied in connection with geophysical fluid dynamics, such as ocean dynamics and atmospheric dynamics. But in compressible fluids, the density stratification under the gravity is not enough and the entropy stratification is essential.

In this report, selective withdrawal phenomenon[6], which is a typical phenomenon in stratified flow, according to the lattice BGK compressible fluid model is simulated, and the effectiveness of the model is confirmed.

2. The thermal lattice BGK model

2.1 The thermal lattice BGK equation

The discrete lattice BGK equation is[3-4]

$$\frac{\partial f_i}{\partial t} + c_i \nabla f_i = -\frac{1}{\phi}(f_i - f_i^{(0)}) \tag{1}$$

where f_i is the particle distribution function in i direction, $f_i^{(0)}$ refers to the local equilibrium distribution function, c_i is the particle velocity, and ϕ is the relaxation parameter.

The dynamics of the fluid can be described by the distribution function obeying the lattice BGK equation (1) and the macroscopic variables are given by the equilibrium distribution function.

Here, the fundamental physical variables are the density ρ, the momentum ρu_α, and the internal energy e

$$\rho = \sum_{\sigma,i} f_{\sigma i} = \sum_{\sigma,i} f_{\sigma i}^{(0)}, \tag{2}$$

$$\rho u_\alpha = \sum_{\sigma,i} f_{\sigma i} c_{\sigma i\alpha} = \sum_{\sigma,i} f_{\sigma i}^{(0)} c_{\sigma i\alpha}, \tag{3}$$

$$\rho e = \sum_{\sigma,i} \frac{1}{2} f_{\sigma i} c_{\sigma i\alpha}^2 - \frac{1}{2}\rho u^2 = \sum_{\sigma,i} \frac{1}{2} f_{\sigma i}^{(0)} c_{\sigma i\alpha}^2 - \frac{1}{2}\rho u^2, \tag{4}$$

where u_α is the component of flow velocity and the index α represents Cartensian coordinates.

The local equilibrium distribution function in equation (1) is expressed as

$$f_{\sigma i}^{(0)} = F_\sigma \rho[1 - 2Bc_{\sigma i\alpha}u_\alpha + 2B^2 c_{\sigma i\alpha} c_{\sigma i\beta} u_\alpha u_\beta + Bu^2 - 2B^2 c_{\sigma i\alpha} u_\alpha u^2 - \frac{4}{3} B^3 c_{\sigma i\alpha} c_{\sigma i\beta} c_{\sigma i\gamma} u_\alpha u_\beta u_\gamma] \tag{5}$$

The moving particles are allowed to move with five kinds of speed, c, $2c$, $3c$, $\sqrt{2}c$, and $2\sqrt{2}c$. Here, the velocity of particles is determined as

$$c_{mki} = k\sqrt{m}c\left\{\cos\left(\frac{\pi(i-1)}{2} + \frac{\pi(m-1)}{4}\right), \sin\left(\frac{\pi(i-1)}{2} + \frac{\pi(m-1)}{4}\right)\right\} \tag{6}$$

$$(i=1,\ldots,4,\ m=1,2,\ k=1,2,\ldots)$$

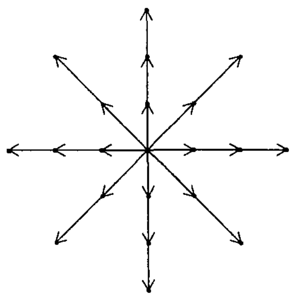


Fig.1 A copmressible lattice Boltzmann model

where σ in equation (5) shows m, k in the equation (6), and $m=1$ is the particle which moves the square edge is paralleled, and $m=2$ is correspondent with the particle which moves in the diagonal line direction, and k shows the speed of integer twice of the speed of the particle moved to the nearest neighbor lattice.

The Navier-Stokes equations are derived from the above equation and the Chapman - Enskog expansion. The function F_0, F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, are determined as

$$F_0=1+\frac{5}{4Bc^2}\left(\frac{17}{96B^2c^4}+\frac{35}{48Bc^2}+\frac{49}{45}\right) \quad (7a)$$

$$F_{11}=-\frac{1}{8Bc^2}\left(\frac{13}{16B^2c^4}+\frac{71}{24Bc^2}+3\right) \quad (7b)$$

$$F_{12}=\frac{1}{16Bc^2}\left(\frac{5}{16B^2c^4}+\frac{25}{24Bc^2}+\frac{3}{5}\right) \quad (7c)$$

$$F_{13}=-\frac{1}{24Bc^2}\left(\frac{1}{16B^2c^4}+\frac{1}{8Bc^2}+\frac{1}{15}\right) \quad (7d)$$

$$F_{21}=\frac{1}{4B^3c^6}\left(\frac{Bc^2}{3}+\frac{1}{8}\right) \quad (7e)$$

$$F_{22}=-\frac{1}{1536B^3c^6}(2Bc^2+3) \quad (7f)$$

and, $B=-\dfrac{1}{2e}$. (7g)

2.2 Introduction of the gravitational force

We shall consider the external force, especially the gravitational force. When the gravitational force acts downward, the density, the momentum, and the internal energy will be changed as

Mass : $\rho \rightarrow \rho$ (8)

Momentum : $\rho u \rightarrow \rho(u-g\cdot\phi)$ (9)

Internal Energy :

$$\frac{1}{2}\rho u^2+\rho e \rightarrow \frac{1}{2}\rho u^2+\rho e-\frac{1}{2}\rho|g|^2\cdot\phi. \quad (10)$$

And the equations (8), (9), and (10) substituting in equation (5) and (7), and including the effect of the gravity in the equilibrium distribution function, it is chosen that this is written with $f^{(0)g}_{pki}$.

Then equation (1) is written as

$$\frac{\partial f_i}{\partial t}+c_i\nabla f_i=-\frac{1}{\phi}(f_i-f_i^{(0)g}) \quad (11)$$

and by using the Chapmann-Enskog expansion for the equation (11), and taking the moment of c_i, the Navier-Stokes equation as the gravitational force works is obtained by

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial r_{1\alpha}}(\rho u_\alpha)=0 \quad (12)$$

$$\frac{\partial}{\partial t}(\rho u_\alpha)+\frac{\partial}{\partial r_{1\beta}}(\rho u_\alpha u_\beta)=-\frac{\partial P}{\partial r_{1\alpha}}-\rho g_\alpha$$
$$+\frac{\partial}{\partial r_{1\beta}}\mu\left(\frac{\partial u_\beta}{\partial r_{1\alpha}}+\frac{\partial u_\alpha}{\partial r_{1\beta}}\right)+\frac{\partial}{\partial r_{1\alpha}}\left(\lambda\frac{\partial u_\gamma}{\partial r_{1\gamma}}\right) \quad (13)$$

$$\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u^2\right)+\frac{\partial}{\partial r_{1\alpha}}\left(\rho e+P+\frac{1}{2}\rho u^2\right)u_\alpha-\rho g_\alpha u_\alpha$$
$$=\frac{\partial}{\partial r_{1\alpha}}\left(k^*\frac{\partial e}{\partial r_{1\alpha}}\right)+\frac{\partial}{\partial r_{1\alpha}}\left\{\mu u_\beta\left(\frac{\partial u_\beta}{\partial r_{1\alpha}}+\frac{\partial u_\alpha}{\partial r_{1\beta}}\right)\right\}$$
$$+\frac{\partial}{\partial r_{1\alpha}}\left(\lambda\frac{\partial u_\beta}{\partial r_{1\beta}}u_\alpha\right) \quad (14)$$

The pressure, kinetic viscosity, the second viscosity, and the conductivity of the internal energy of this fluid are given, respectively, by

$$P=\frac{2}{D}\rho e,\quad \mu=\frac{2}{D}\rho e\tau\left(\phi-\frac{1}{2}\right)$$
$$\lambda=-\frac{4}{D^2}\rho e\tau\left(\phi-\frac{1}{2}\right),$$
$$k^*=\frac{2(D+2)}{D^2}\rho e\tau\left(\phi-\frac{1}{2}\right) \quad (15a,b,c,d)$$

where D indicates the dimension and 2 in this case.

2.3 Definition of the entropy

In the two-dimensional mode, the pressure p and the entropy s are obtained, respectively, by

$$p=e\rho,\ \text{and}\ s=c\log\left(\frac{p^{1/\gamma}}{\rho}\right) \quad (16),(17)$$

where the ration of the specific heats γ is given by $(D+2)/D$, and in two-dimensional flows $\gamma=2$. Therefore we have

$$s \propto \log e-\log\rho. \quad (18)$$

2.4 The potential density

In the concept often used in geophysical fluid dynamics, there is that of the potential density [7].

Though this is a concept equal to the entropy stratification, when the fluid motion is isentropic, the density as some given density distribution changes in one standard pressure fluid isentropic like what was said, and when this potential density decreases for the upper part, the fluid is stable, and it is unstable when it is reversed.

That is to say, potential density distribution is correspondent to the density distribution in the incompressible fluid. Here, we consider that the Froude number Fr is defined using the density in the channel of upper and lower side.

To begin with, the amount as there is no disturbing in the flow shall be shown in suffix B. At this time, the potential density is expressed as

$$\rho_B = \rho_0 \left(\frac{p_0}{p_B} \right)^{1/r}, \tag{19}$$

where suffix 0 shows the standard position, and the ratio of specific heats is $r = 2$ in the two-dimensional model.

The standard position is taken at lower side in the channel, and it is shown in suffix 1, 2, respectively, and the buoyancy frequency N and the Froude number of the flow are defined, respectively, as

$$N = \left(-\frac{g}{d} \frac{\rho_{B1} - \rho_{B2}}{\rho_{B1}} \right)^{1/2} \tag{20}$$

$$Fr = \frac{Q}{Nd^2}. \tag{21}$$

Here, d is the channel height and Q is the fluid volume flow rate which flows out the sink in unit time.

3. Line sink flow and calculation

Line sink flow in the stratified flow as shown in Fig. 2 is calculated, and we examined degree of stratification of the entropy and aspect of the sink flow. The fluid the gravitational force acts downward, and it flows

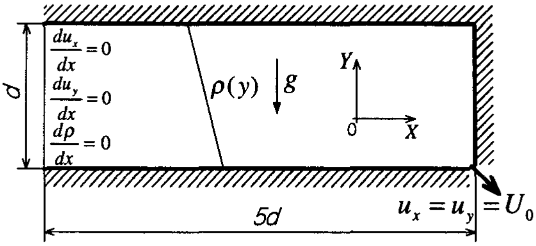


Fig. 2 Two-dimensional line sink flow

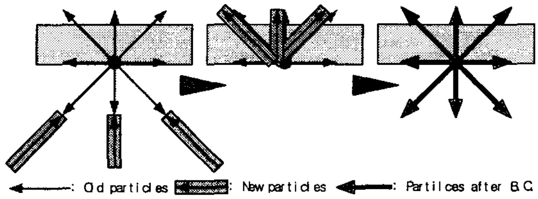


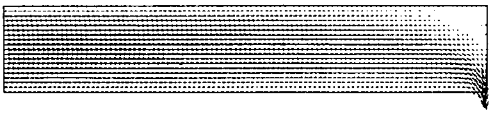
Fig. 3 Boundary condition at solid wall

in the channel which established solid walls in which there are level for top and bottom and solid wall which is perpendicular to right side as shown in Fig.2. The left side in the region extends infinitely. The stratified fluid in this region drain out the line sink which locates at the right corner starting from quiescent state.

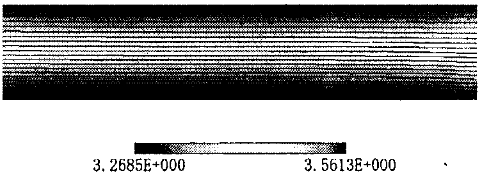
On the solid wall, the boundary condition is diffusive reflection of the particle. Particles which reach solid wall rebound in a velocity distribution given in equilibrium distribution function (5) which depends on the speed, the density, and the temperature (the internal energy). In short, it is correspondent with giving no-slip condition and condition of the temperature, in the solid wall. The density in the boundary is determined as follows: the particles which approach the boundary stops, as it is shown in Fig. 3, and they are added the particles which flow into the boundary (central figure), and the density in the boundary node is so defined.

Equation (1) is solved by the second order Runge-Kutta method, and the space derivative of convective term is discretized by the QUICK.

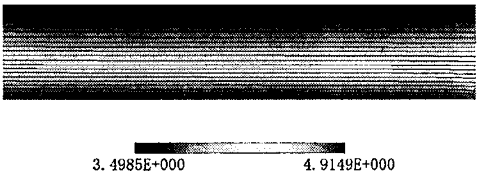
The velocity of the sink is set to be 0.02 in



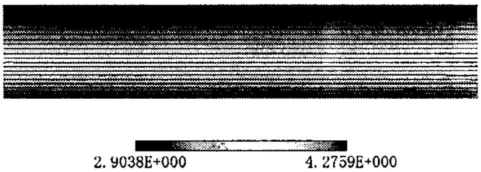
(a) Velocity vectors



(b) Entropy



(c) Density



(d) Pressure

Fig. 4 Uniform entropy

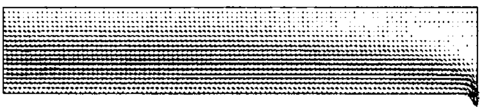


Fig. 5 Velocity vectors in stable stratification

all calculations, and the width of the sink is made to be the 3 lattices. We set the acceleration of gravity $g=0.02$, and $\triangle t=0.01$ for time, and the relaxation coefficient is being changed by the calculation. That is to say, it is chosen so that the calculation may stably advance by changing the viscosity coefficient.

The density was uniformly set to be 1.0 as an initial condition in all regions, and the internal energy is also uniformly made to be mean value of upper wall and lower wall. The calculation is carried out without sinking, and the sink started, after it almost reached the steady state.

The fluid of the lower level receives the compression, when the gravity works, and the temperature (the internal energy) increases. Then, by the energy diffusion (by the temperature conduction), the internal energy is carried above, so that the entropy of the upper part increases, and the nonuniformity of the entropy occurs. Therefore, we shall generate the entropy stratification by adequately changing the boundary condition of internal energy of upper wall and lower wall, without setting.

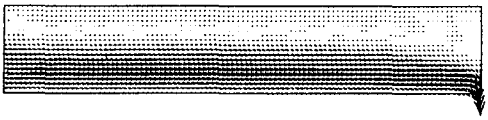
4. Results and discussion

The calculation is carried out as an initial-value problem, as it is described in Chapter 3, but from now, we just consider the result when the calculation almost reach steady state.

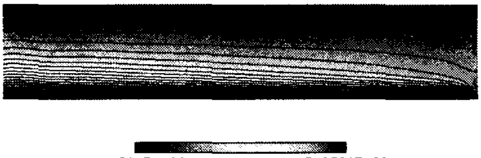
Figure 4 shows the velocity vectors in almost uniformly distributing of the entropy in the flow field. The internal energy on the top wall $e_2=0.83$, and that on the bottom wall $e_1=0.87$.

Though the density increases for the downward, as is shown in Fig. 4(c) under the effect of the gravity, the entropy is almost uniform, as is shown at Fig. 4(b), and the flow is in a condition of a neutral stability. The fluid flows out to the sink from the whole region this time. This is correspondent with the flow as the density is uniform. Figure 4(d) also shows the pressure distribution.

Though the stratification is stable in Fig. 5, the velocity vector distribution as the intensity of the stratification is not sufficient, is shown. The temperature in the top wall and the bottom wall is $e_1=0.5$, $e_2=1.0$, respectively.

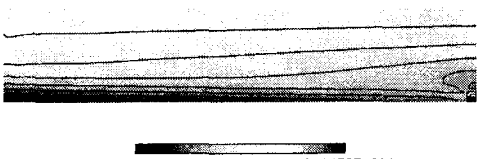


(a) Velocity vectors



(b) Entropy

(c) Density



(d) Pressure

Fig. 6 Strong stable stratification

In this case, a dividing stream line appears, and stagnation area occurs for the upper part of the sink, but the fluid from the whole region flows into the sink.

When the Froude number defined in Chapter 2 is considered, it becomes 0.4×10^{-2} with very small value. For the flow of this small Froude number, only the flow in the horizontal leve almost equal to the sink is selectively flows out in case of the inviscid incompressible flow, but, in present calculation, the fluid from the whole region flows out.

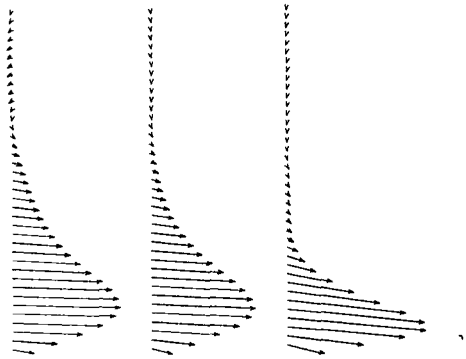
The Reynolds number is 50 calculated from the flowrate and the coefficient of kinematic viscosity, so that the effect of the viscosity seems big. Here, we can suppose that it is possible to consider the effect of the compressibility. In the vicinity of the sink point, there is the expansion of fluid.

In the incompressible fluid, the vorticity occurs from the deviation of pressure gradient and density gradient, and it is the reason why this selective withdrawal is established in the flow, whereas in the fluid expands, it propagates almost isotropically on the effect of sink as expansion wave, and it substantially has the effect in which mouth of sink expands.

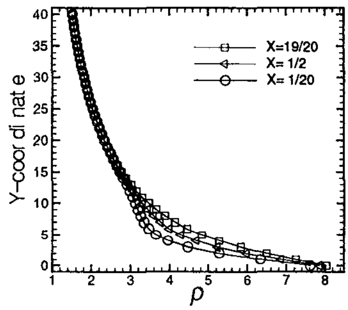
And, the pressure as a fluid which close to the bottom wall is high, the proportion of the density reduction by the expansion is big. In short, the pressure drop will be considerable, and the robust fluid flows into the sink in the effect of the gravity and seems to generate not only horizontal flow but also the flow which approaches the stock. This is remarkable in case of following the strongly stable stratification.

The strong entropy stratification is obtained, when the temperature of the top wall and the bottom wall is put $e_1=0.5$, $e_2=1.5$, respectively. In case of the stratification which is sufficiently stable like this, the fluid of lower half is withdrawn which proven from velocity vector distribution shown in Fig. 6(a), and there is making circulating flows at the top region. It is a phenomenon equal to selective withdrawal phenomenon observed in the reservoirs which are established the temperature stratification. Figure 6(b), (c) and (d) show the distribution of the entropy, the density and the pressure, respectively. In the pressure distribution of Fig. 6(d), there is the rapid decreasing of the pressure near the sink, as stated above it.

In the sufficiently stable stratification, such selective flow is generated the internal gravity wave from the line sink, and the mode is formed by the top and bottom solid walls, and that it propagates to the upstream[8]. In the incompressible inviscid fluid, the pattern of the

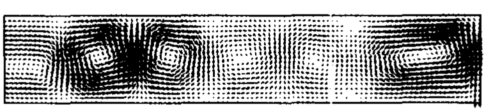


(a) Velocity vectors

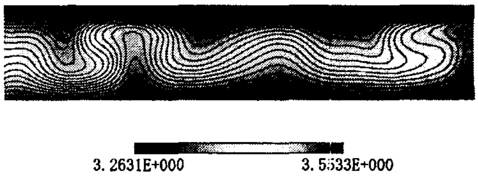


(b) Density distribution

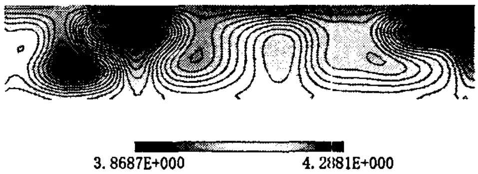
Fig. 7 Velocity profiles and density at x=19/20, 1/2 and 1/20



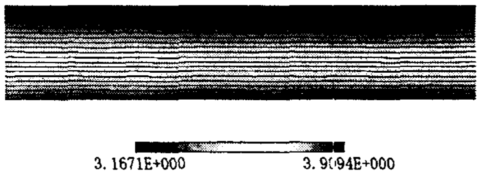
(a) Velocity vectors



(b) Entropy



(c) Density



(d) Pressure

Fig. 8 Unstable stratification

clear internal gravity wave is obtained for the flow of low Froude number. But in present calculation, we can't get the wave pattern. Though the lattice number is taken for each double 200×40 in length and height direction on trial, the calculation result is equal to the pattern as the lattice is fewer. It is considered that, in short, the pattern of the clear internal wave is not obtained in this calculation. Though the effect of the viscosity is also considered, as it is mentioned as this reason earlier, the effect of the compressibility seems to be also big.

In order to examine the compressibilty effect in the flow in detail, the velocity distribution and the density distribution at 3 cross section in the examined channel when the flow is almost steady. The result is respectively shown in Fig. 7(a) and (b). Here, from the left in the figure, the distances from the sink are positions each 19/20, 1/2 and 1/20 for the channel length. We obtain that the mass flow rate in each cross section is an almost equal value, and the continuity is satisfied.

However, when density distribution is observed, the density decreases, as it goes near 5 to the downstream in lattice node from the wall of the lower. And, at 1/20 cross section, it is turned to the flow velocity vector, because of the sink and fluid expansion effect, incline to downward but it is almost horizontal. The density is a function only of the streamline, if the flow is the incompressible fluid, because it is almost steady.

In figure 8, the flow of the unstable

stratification is shown. The temperature in the top wall and the bottom wall is put $e_1=0.75$, $e_2=0.95$, respectively. The convection pattern like Benard convection appears from the flow distribution (a), the entropy distribution (b), the density distribution (c). Then, the effect of the sink is hidden for large fluctuation of the convection pattern due to this instability. In the meantime, in the pressure distribution (d), the iso-pressure line is almost horizontal, and it is proven to be a flow field which establishes Boussinesq approximation.

5. Conclusion

By the finite difference calculation using compressible lattice Boltzmann fluid model, sink flow in the two-dimensional entropy stratification was calculated.

And the difference between the inviscidity and the incompressible fluid in sink flow is also examined from the viewpoint of the compressibility.

References

[1] Rotheman, D. H. and Zaleski, S., "Lattice-Gas Cellular Automata, Cambridge Up (1997).
[2] Qian, Y. H., Succi, S. and Orszak, S. A., "Recent Advances in Lattice Boltzmann Computing," Ann. Rev. of Comp. Phy. III, D. Staufer ed. World Sci. Pub (1995).
[3] Cao, N., Chen, S., Jin, S. and Martinez, D., Phys. Rev. E, 55 (1997)
[4] 瀨田剛외 3 명, 日本機械學會論B . 65-634 (1997), p.1955
[5]高田尙樹、蔦原通久、日本機械學會論B . 64-628 (1998) 3934
[6] Yih C. S., Stratified Flows, Academic (1980)
[7]木村龍二、地球流體力學入門、東京堂出版 (1983)
[8]片岡武、蔦原通久、日本流體力學會誌「ながれ」 (1997) 248.