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HIGHER ORDER SINGULARITIES AND THEIR ENERGETICS IN ELASTIC-PLASTIC FRACTURE

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탄소성 균열 문제에서 고차응력특이성과 에너지론

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Abstract

The higher order singularities[1] are systematically examined, and discussed are their complementarity relation with the nonsingular eigenfunctions and their relations to the configurational forces like J-integral and M-integral. By use of the so-called two state conservation laws(Im and Kim[2]) or interaction energy, originally proposed by Eshelby[3] and later treated by Chen and Shield[4], the intensities of the higher order singularities are calculated, and their roles in elastic-plastic fracture are investigated. Numerical examples are presented for illustration.

1. Introduction

The purpose of the present work is to review the complementarity relationships among the eigenvalues in an eigenfunction expansion for a generic isotropic wedge together with our recent applications to wedge and crack problems [2,5,6,7]. Furthermore, we demonstrate their application for characterizing elastic-plastic cracks via high order singularities and their complementary eigenpairs, which are nonsingular terms.

2. Governing Equations and *J*- and *M*-Integral

For the plane problem, the J-integral and the M-integral[2] may be written as:

$$J = \int_\Gamma (Wn_1 - t_i u_{i,1})ds \tag{1}$$

$$M = \int_\Gamma (Wn_l - t_i u_{i,l})x_l ds \tag{2}$$

where n_l is the component of unit outward normal on the contour Γ; W and t_i indicate the strain energy density and the traction component, given as $W = C_{ijkm}\varepsilon_{ij}\varepsilon_{km}/2$ and $t_i = \sigma_{ij}n_j$.

Consider two independent elastic states "A" and "B" for the plane strain problems. We suppose another elastic state "C", which is obtained by superposing the two equilibrium states "A" and "B". Then the path-independent integrals J and M are written as

$$J^C = J^A + J^B + J^{(A,B)} \tag{3}$$

$$M^C = M^A + M^B + M^{(A,B)} \tag{4}$$

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where the superscripts "A", "B" and "C" indicate the aforementioned elastic states, and $J^{(A,B)}$ and $M^{(A,B)}$ are given as

$$M^{(A,B)} = \int_C [C_{ijkm}\varepsilon_{ij}^A \varepsilon_{km}^B n_n - (t_i^A \frac{\partial u_i^B}{\partial x_n} + t_i^B \frac{\partial u_i^A}{\partial x_n})] x_n ds \quad (5)$$

$$J^{(A,B)} = \int_C [C_{ijkm}\varepsilon_{ij}^A \varepsilon_{km}^B n_1 - (t_j^A \frac{\partial u_j^B}{\partial x_1} + t_j^B \frac{\partial u_j^A}{\partial x_1})] ds \quad (6)$$

The integrals $J^{(A,B)}$ and $M^{(A,B)}$ result from the mutual interaction between two elastic states "A" and "B". These are conservation integrals for two equilibrium states, since the area integral version of these contour integrals vanishes identically for the domains with no singularities

We briefly summarize the structure of the asymptotic solutions in the form of eigenfunction series for the two-dimensional wedge problem. The eigenfunction expansion for the stress and displacement components for the generic wedge may be written in the following power type function of $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ [2]:

$$\sigma_{\alpha\beta}^{(m)} = Re[\sum_{\delta_n} \beta_n \sum_{k=1}^{2} \{ C_{kn}^{(m)} (\Lambda_{\alpha\beta k} g_n'(z) + \Gamma_{\alpha\beta k} \bar{z} g_n''(z)) + C_{(k+2)n}^{(m)} (\bar{\Lambda}_{\alpha\beta k} g_n'(\bar{z}) + \bar{\Gamma}_{\alpha\beta k} z g_n''(\bar{z}))\}] \quad (7)$$

$$u_\alpha^{(m)} = \frac{1}{2\mu^{(m)}} Re[\sum_{\delta_n} \beta_n \sum_{k=1}^{2} \{ C_{kn}^{(m)} (p_{\alpha k}^{(m)} g_n(z) + q_{\alpha k} \bar{z} g_n'(z)) + C_{(k+2)n}^{(m)} (\bar{p}_{\alpha k}^{(m)} g_n(\bar{z}) + \bar{q}_{\alpha k} z g_n'(\bar{z}))\}] \quad (8)$$

with $g_n'(z) = z^{\delta_n}$ and μ being the shear modulus; the non zero components of $\Lambda_{\alpha\beta k}$, $\Gamma_{\alpha\beta k}$, $p_{\alpha k}$ and $q_{\alpha k}$:

$$-\Lambda_{111} = \Lambda_{221} = i\Lambda_{121} = 1,\ \Lambda_{112} = \Lambda_{222} = 2,$$
$$\Gamma_{112} = -\Gamma_{222} = -i\Gamma_{122} = -1, p_{11}^{(m)} = -ip_{21}^{(m)} = -1, \quad (9)$$
$$p_{12}^{(m)} = ip_{22}^{(m)} = 3 - 4\nu^{(m)},\ q_{12} = -iq_{22} = -1$$

where δ_n is an eigenvalue and ν is Poisson's ratio; C_{kn}, short for $C_k(\delta_n)$, is the corresponding eigenvector; $\beta_n = \beta(\delta_n)$ represents the load parameter or the intensity of the elastic field associated with eigenvalue δ_n. Note that β_n is real for a real δ_n, but it is, in general, complex for a complex δ_n. For a complex δ_n, it is self-evident from the expression (10) that its conjugate $\bar{\delta}_n$ also belongs to the eigenvalues. For clarity, we assume that the imaginary part of complex δ_n is positive in the expression (10) since a complex eigenvalue δ_n and its conjugate $\bar{\delta}_n$ lead to the same eigenfunction. The superscript "*(m)*" indicates the *m*-th sector.

For an arbitrary eigenvalue δ_l in equations (7) and (8), we first define its complementary eigenvalue δ_l^c in the *M*-integral sense such that

$$\delta_l^c + \delta_l = -2 \quad (10)$$

It has been shown that δ_l^c is also an eigenvalue for a given wedge [2, 8]. For cracks around which the *J*-integral becomes path-independent, in a similar way we can define the complementary eigenvalue δ_l^c in the *J*-integral sense for an arbitrary eigenvalue δ_l:

$$\delta_l^c + \delta_l = -1 \quad (11)$$

Equation (10) can be utilized for finding the intensities of the singular terms in the eigenfunction expansion for the re-entrant vertices of thin films [2] and adhesive lap-joints [5]. Recently it is shown that equation

(11) together with (10) is useful for decomposing local three dimensional crack tip field under mixed mode to obtain the individual stress intensities [6]. In addition to the recent results of Ref. [2,5,6], in the present presentation we will show that equation (10) or (11) may be applied for computing the intensities of the high order singularities [1] and the nonsingular terms as well for elastic-plastic crack tips. This enables us to characterize the elastic-plastic cracks via these additional terms together with the inverse square root singularity. Specializing equation (7) for elastic-plastic cracks with plastic zone removed, we obtain the expressions:

$$\begin{aligned}\sigma_{ij} &= \cdots + \beta_{-2} r^{-3/2} f_{ij}(\theta,-3/2) + \beta_{-1} r^{-1} f_{ij}(\theta,-1) \\ &+ (K_I/\sqrt{2\pi})r^{-1/2} f_{ij}(\theta,-1/2) + \beta_1 \delta_{1i}\delta_{1j} \\ &+ \beta_2 r^{1/2} f_{ij}(\theta,1/2) + \beta_3 r f_{ij}(\theta,1) + \cdots \qquad (12)\end{aligned}$$

where K_I and β_1 represent the stress intensity factor and the *T*-stress, respectively. Utilizing the aforementioned two-state *J*- or *M*-integral with the aid of finite element analysis, we calculate the intensities β_j as well as K_I for elastic-plastic crack tip. The result shows that the contribution to *J*-integral comes from each complementary pair of eigenfunctions, defined by equation (11), in addition to the inverse square root singularity, whose complementary pair is itself or $\delta_0 = \delta_0^c = -1/2$.

For the elastic-plastic plane problem, the *J*-integral and *M*-integral can be written as

$$J = J_0 + \sum_{n=1}^{\infty} J^{(\delta_n,\delta_n^c)} \qquad (13)$$

$$M = \sum_{n=0}^{\infty} M^{(\delta_n,\delta_n^c)} \qquad (14)$$

where $J^{(\delta_n,\delta_n^c)}$ and $M^{(\delta_n,\delta_n^c)}$ are the two-state conservation integrals[2], which provide the method for calculating the intensities of higher order singularities. For the eigenvalue δ_n in plane strain crack problems, we have complementary eigenvalue $\delta_n^c = -1 - \delta_n$ in the *J*-integral sense and $\delta_n^c = -2 - \delta_n$ in the *M*-integral sense wherein $J^{(\delta_n,\delta_n^c)}$ and $M^{(\delta_n,\delta_n^c)}$ retain the path independence. $J_0 = J(-1/2,-1/2)$ is the classical expression of the *J*-integral for elastic crack problem calculated with $\delta_0 = -1/2$. The summation $\sum_{n=1}^{\infty} J^{(\delta_n,\delta_n^c)}$ and $\sum_{n=0}^{\infty} M^{(\delta_n,\delta_n^c)}$ are the *J*-integral and the *M*-integral contribution due to each of the eigenvalues δ_n and their conjugate eigenvalues δ_n^c, respectively. J_0, related to the translation of $\delta_0 = -1/2$ singularity, means the extension of the crack, and the remains terms $\sum_{n=1}^{\infty} J^{(\delta_n,\delta_n^c)}$ are associated with translation of the higher order singularities. Furthermore $\sum_{n=0}^{\infty} M^{(\delta_n,\delta_n^c)}$ is identically zero in the absence of the plastic zone, but increases in proportion to the size of the plastic zone due to the increasing influence of the higher order singularities.

3. Numerical Results and Conclusion

For a numerical example, we choose a homogeneous isotropic material with the properties: $E = 71GPa$, $\nu = 0.33$ and $\sigma_Y = 303MPa$. For the model of this study, we select single edge notched tension(SENT) panel(see Fig. 1) and the Mode I loading is applied on the boundary of the model(see Fig. 2). The calculation was processed with a power-law hardening material in a uniaxial stress-strain law that is the plastic part of the Ramberg-Osgood formula. We choose $m = 10$ and $\alpha = 1.0$ for numerical computation. The package code ABAQUS is used for the finite element solution. Fig. 3 and 4 show that the computed J, J_0 and each value of $J^{(\delta_n,\delta_n^c)}$ as well as M and $M^{(\delta_n,\delta_n^c)}$ increase as the maximum plastic zone size increases. From these figures we may draw the conclusion that $J^{(1/2,-3/2)}$ and $M^{(-1/2,-3/2)}$ have the most dominant effect on $\sum_{n=1}^{\infty} J^{(\delta_n,\delta_n^c)}$ and $\sum_{n=0}^{\infty} M^{(\delta_n,\delta_n^c)}$, respectively.

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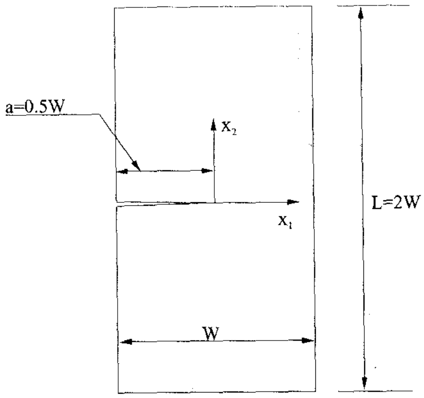

Fig. 1 The SENT panel under plane strain

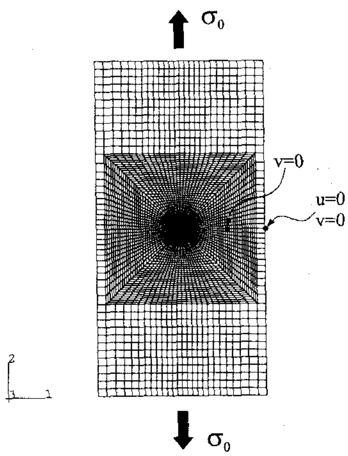

Fig. 2 F.E. mesh and boundary condition

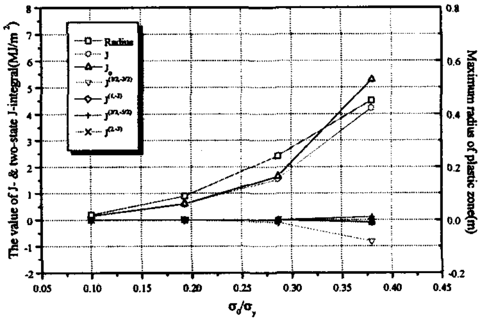

Fig. 3 Maximum radius of plastic zone and computed *J*-integral and two-state *J*-integrals versus applied loading

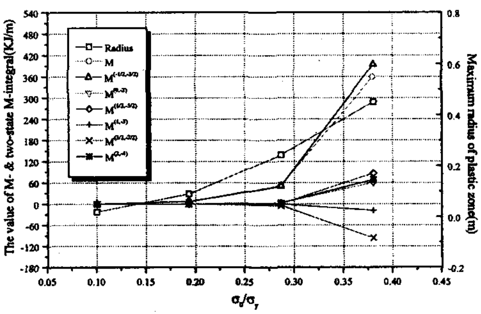

Fig. 4 Maximum radius of plastic zone and computed *M*- and two-state *M*-integrals versus applied loading