# Master-Slave Synchronization of Bifurcating Integrate-and-Fire Circuits

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## 1. Abstract

We consider a master-slave pulse-coupled network of bifurcating integrate-and-fire circuits. The network exhibits in-phase chaotic synchronization and various periodic synchronization phenomena. In order to analyze these phenomena precisely, we derive a one-dimensional return map. Also using a simple test circuit, typical phenomena are demonstrated in the laboratory.

## 2. Introduction

In this paper we consider synchronization phenomena in a master-slave pulse-coupled network of bifurcating integrate-and-fire circuits. As a state of the master/slave reaches a firing threshold, the state is reset to a periodic base level and a pulse is generated. This is called the self-firing. The output pulse-train of the master is input to the slave. If an input pulse arrives and the slave state is above a refractory threshold, the slave fires. This is called the compulsory-firing. Due to interactions of self-firings and compulsory-firings, the network exhibits various synchronization phenomena as the following. Let the slave be periodic before the coupling. If the master is periodic, the slave exhibits compulsory-firings and self-firings and the network exhibits various periodic synchronizations. Let the slave be chaotic before the coupling. If the master is chaotic, the slave exhibits compulsory-firings and the network exhibits chaotic synchronization. If the master is periodic, the network can also exhibit various periodic synchornizations. In order to analyze these phenomena exactly, we derive a one-dimensional return map for the compulsory-firing moments. We also characterize the phenomena using a rate of compulsory-firings and self-firings. Typical phenomena are demonstrated in the laboratory using a simple test circuit.

Integrate-and-fire circuits have been studied as simplified neuron models [1][2], and their pulse-coupled neural networks (PCNNs) have been constructed. Study of synchronization phenomena of the PCNN is an important nonlinear problem, and many fundamental results have been published [3][4]. Based on periodic synchronization, applications of PCNNs have been considered, e.g., associative memory, image segmentation and combinational optimization problem solver [5]-[7]. However chaotic or various periodic synchronizations of PCNNs have not been considered sufficiently so far. Hence our results may contribute to develop a PCNN having some flexible function based on various synchronization phenomena. Basic results can be found in [8].

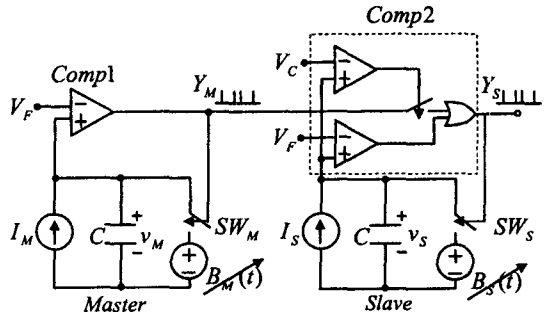


Figure 1. Master-slave pulse-coupled network.

## 3. Master-Slave pulse-coupled network

We consider a master-slave pulse coupled network of bifurcating integrate-and-fire circuits as shown in Fig.1. Base signals $B_M(t) = -K_M \sin(2\pi t/T)$ of the master and $B_S(t) = -K_S \sin(2\pi t/T)$ of the slave have period $T$ and are synchronized. When the master state $v_M$ reaches a firing threshold $V_F$, the master fires: a firing pulse $Y_M = V_H$ is generated, the switch $SW_M$ is closed, and the state $v_M$ is reset to the base $B_M(t)$. When the slave state $v_S$ reaches a firing threshold $V_F$, the slave fires in a likewise master. In addition, if $V_C < v_S < V_F$ and an input pulse $Y_M = V_H$ arrives, the slave fires. Using the following dimensionless variables and parameters

$$\begin{aligned} x_M &= \tfrac{v_M}{V_F},\ s_M = \tfrac{TI_M}{CV_F},\ k_M = \tfrac{K_M}{V_F},\ \tau = \tfrac{t}{T}, \\ x_S &= \tfrac{v_S}{V_F},\ s_S = \tfrac{TI_S}{CV_F},\ k_S = \tfrac{K_S}{V_F}, \\ y_M &= \tfrac{Y_M - V_L}{V_H - V_L},\ y_S = \tfrac{Y_S - V_L}{V_H - V_L},\ th_C = \tfrac{V_C}{V_F}. \end{aligned} \tag{1}$$

the network dynamics is described by

$$\begin{cases} \dot{x}_M = s_M & \text{for } x_M < 1, \\ x_M(\tau^+) = b_M(\tau^+) & \text{if } x_M(\tau) = 1, \end{cases}$$
$$y_M(\tau^+) = \begin{cases} 0 & \text{for } x_M < 1, \\ 1 & \text{if } x_M(\tau) = 1, \end{cases} \tag{2}$$
$$b_M(\tau) = -k_M \sin(2\pi\tau),$$

$$\begin{cases} \dot{x}_S = s_S & \text{for } x_S < 1, \\ x_S(\tau^+) = b_S(\tau^+) & \text{if } x_S(\tau) = 1, \\ x_S(\tau^+) = b_S(\tau^+) & \text{if } x_S(\tau) > th_C \text{ and} \\ & y_M(\tau) = 1, \end{cases}$$
$$y_S(\tau^+) = \begin{cases} 0 & \text{for } x_S < 1, \\ 1 & \text{if } x_S(\tau) = 1, \\ 1 & \text{if } x_S(\tau) > th_C \text{ and } y_M = 1, \end{cases} \tag{3}$$
$$b_S(\tau) = -k_S \sin(2\pi\tau).$$

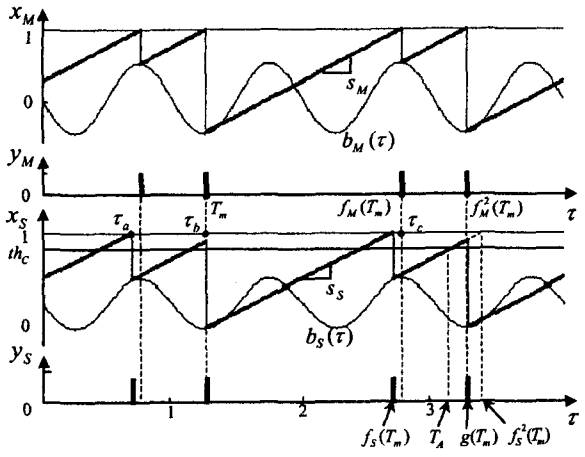

Figure 2. Basic dynamics of the network. $T_A = f_S^2(T_m) - (1 - th_C)/s_S$.

The suffixes "$_M$" and "$_S$" represent master and slave, respectively. $k_M$ and $k_S$ are the amplitudes of the bases $b_M(\tau)$ and $b_S(\tau)$. $s_M$ and $s_S$ are called stimulations. $th_C$ is called refractory threshold. In this paper, we fix $s_M = 1$, $s_S = 0.95$ and $th_C = 0.8$. The controlled parameters are $|k_M| < 1$ and $|k_S| < 1$.

First, we explain the dynamics of the master (see Fig.2). If the state $x_M$ reaches a firing threshold 1, the master outputs the firing pulse $y_M = 1$ and $x_M$ is reset to the base $b_M(\tau)$. This is called the self-firing. Repeating the self-firings, the master generates a pulse-train $y_M(\tau)$. Letting $\tau_n$ be the $n$-th pulse position and letting $\boldsymbol{R}^+$ represent the positive reals, the dynamics of the pulse-train is described by a pulse position map $f_M : \boldsymbol{R}^+ \to \boldsymbol{R}^+$,

$$\tau_{n+1} = f_M(\tau_n) \equiv \tau_n - \frac{1}{s_M} b_M(\tau_n). \tag{4}$$

Noting $f_M(\tau + 1) = f_M(\tau) + 1$, we obtain a return map

$$\tau_{n+1} = F_M(\tau_n) \equiv f_M(\tau_n) \pmod 1. \tag{5}$$

Fig.3 shows examples of the return map $F_M$. We note that the shape of the map $f_M$ is determined by the shape of the base $b_M(\tau)$. Hence we can obtain various pulse-train dynamics by adjusting the shape of the base. Here we introduce the following.

**Definition** A pulse position $\tau^*$ is said to be periodic with period $P$ if $P$ is the minimum integer such that $f^Q(\tau^*) - \tau^* = P$ for some positive integer $Q$, where $f^Q$ denotes the $Q$-fold composition of $f$. A pulse-train $y^*$ is said to be periodic with period $P$ if $y^*$ is represented by the periodic pulse positions $(\tau^*, f(\tau^*), ..., f^{Q-1}(\tau^*))$ and $f^Q(\tau^*) = \tau^* + P$. A periodic pulse-train $y^*$ is said to be stable or unstable if $|Df^Q(\tau^*)| < 1$ or $|Df^Q(\tau^*)| \geq 1$, respectively, where $Df \equiv df/d\tau_n$.

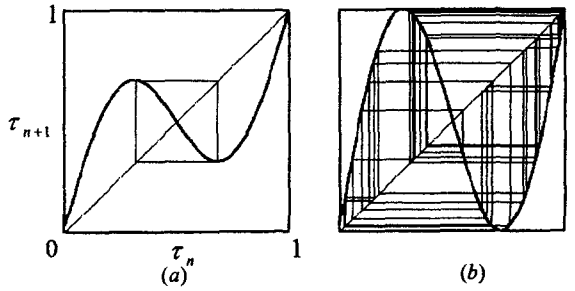

Figure 3. Return map $F_M$. (a) Periodic pulse-train. $k_M = 0.4$. (b) Chaotic pulse-train. $k_M = 0.73$.

Fig.3(a) corresponds to a stable periodic pulse-train, and Fig.3(b) corresponds to a chaotic (non-periodic) pulse-train.

Next, we explain the dynamics of the slave using Fig.2. If the state $x_S$ reaches the firing threshold 1, the slave outputs a pulse $y_S = 1$, and $x_S$ is reset to the base $b_S(\tau)$ as shown at time $\tau_a$. This is the self-firing(SF). If $th_C < x_S < 1$ and an input pulse $y_M = 1$ arrives, the slave outputs a firing pulse $y_S = 1$ and the state $x_S$ is reset to the base $b_S(\tau)$ as shown at time $\tau_b$. This is called the compulsory-firing(CF). If $x_S$ is below the refractory threshold $th_C$, the input pulse does not affect to the slave as shown at time $\tau_c$. Due to the interaction of the CFs and the SFs, the network exhibits various phenomena. Note that because of the CFs, these phenomena become super-stable with respect to the initial state of the slave. Fig.4 shows typical phenomena. Letting $T_m$ be the $m$-th CF moment, the dynamics of the slave is described by the following CF map:

$$T_{m+1} = g(T_m) = f_M^q(T_m), \tag{6}$$

where $q$ is the minimum integer such that

$$f_S^p(T_m) - \frac{1}{s_S}(1 - th_C) < f_M^q(T_m) < f_S^p(T_m), \qquad f_S(\tau) \equiv \tau - \frac{1}{s_S} b_S(\tau), \tag{7}$$

where $p$ is some positive integer. $q$ means the number of the SFs of the master during $T_m \leq \tau \leq T_{m+1}$. $p$ means the number of the SFs of the slave during $T_m \leq \tau \leq T_{m+1}$. We obtain the following CF return map:

$$T_{m+1} = G(T_m) \equiv g(T_m) \pmod 1. \tag{8}$$

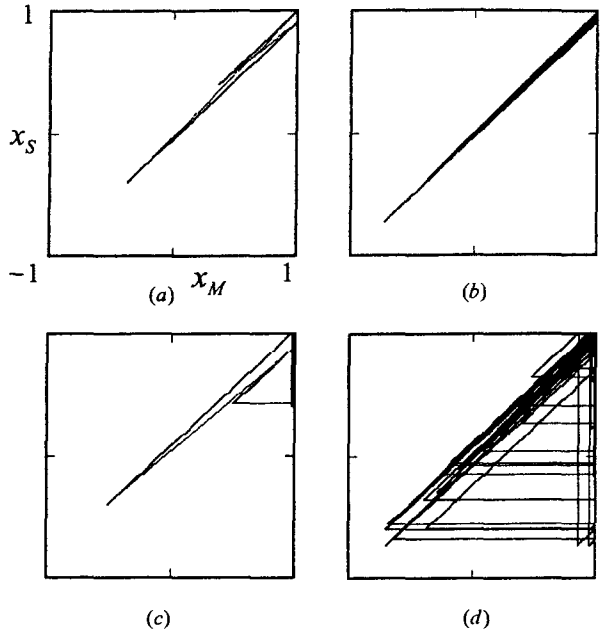


Figure 4. Phase space trajectories. (a)&(c): Periodic slave. $k_S = 0.4$. (b)&(d): Chaotic slave. $k_S = 0.73$. (a) $k_M = 0.4$ and $R_C = 1$. (b) $k_M = 0.73$ and $R_C = 1$. (c) $k_M = 0.5$ and $R_C = 0.5$. (d) $k_M = 0.7$ and $R_C \simeq 0.67$.

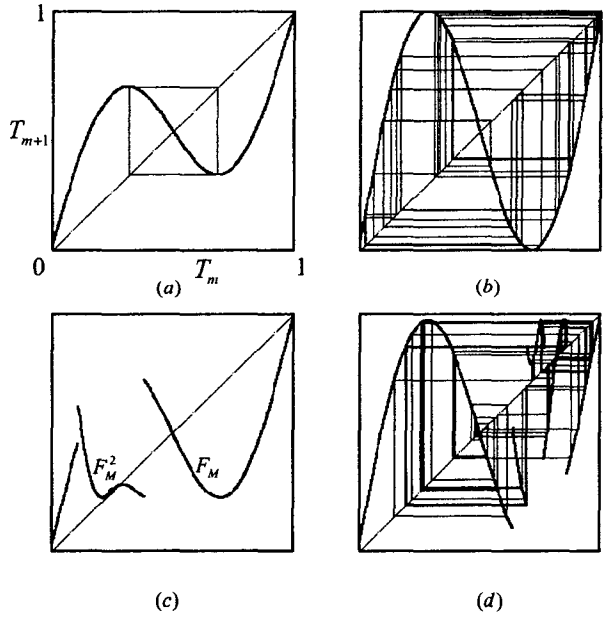


Figure 5. CF return map. (a)&(c): Periodic slave. $k_S = 0.4$. (b)&(d): Chaotic slave. $k_S = 0.73$. (a) $k_M = 0.4$ and $R_C = 1$. (b) $k_M = 0.73$ and $R_C = 1$. (c) $k_M = 0.5$ and $R_C = 0.5$. (d) $k_M = 0.7$ and $R_C \simeq 0.67$. (a)-(d) correspond to Fig.4(a)-(d).

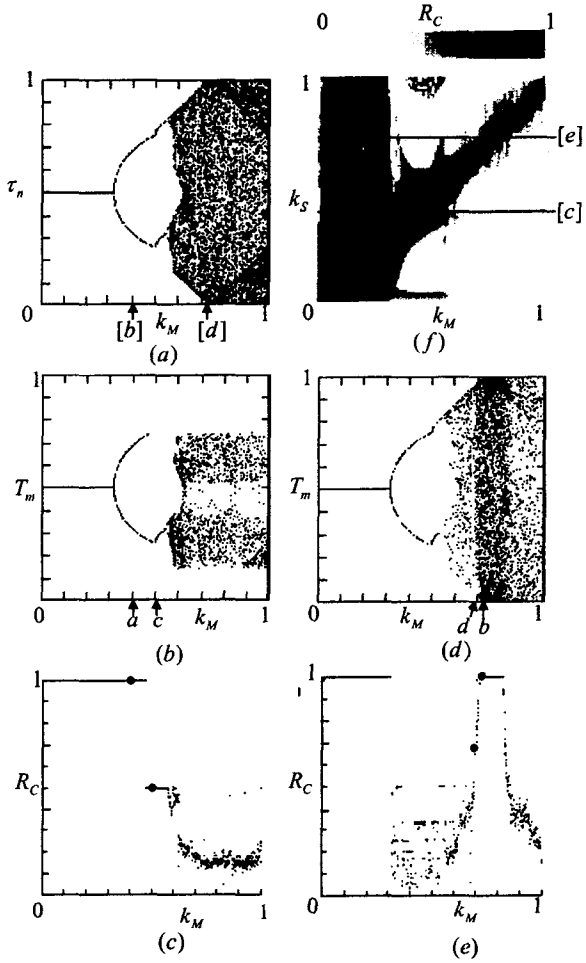


Figure 6. (a) Bifurcation diagram of SF moment $\tau_n$ of the master. (b)&(d) Bifurcation diagram of CF moment $T_m$ of the slave. (b)$k_S = 0.4$. (d)$k_S = 0.73$. (c),(e)&(f) CF rate $R_C$. (c)$k_S = 0.4$. (e)$k_S = 0.73$. [b]-[e] correspond to (b)-(e). *a-d* correspond to Fig.4(a)-(d).

Fig.5 shows CF return map. In order to characterize these phenomena, we introduce a CF rate $R_C$ as the following:

$$R_C \equiv \frac{N_C}{N_S + N_C}, \tag{9}$$

where $N_S$ is the number of SFs, $N_C$ is the number of CFs, and $N_C$ is sufficiently large. We can calculate $R_C$ using the CF map $g$: the iteration number gives $N_C$, and the summation of $p$ in Equ.(7) for each $T_m$ gives $N_S + N_C$.

Fig.6(a) shows bifurcation diagram of the SF moment $\tau_n$ of the master. Let us fix the slave parameter at $k_S = 0.4$. In this case, if the slave has no pulse-train input $y_M$, it is periodic. Fig.6(b) and (c) show a bifurcation diagram of the CF moment $T_m$ and the CF rate $R_C$ for the master parameter $k_M$. If $R_C = 1$, the slave exhibits the CF only, and $T_m = \tau_n$. Then, the plots of the CF moment $T_m$ are identical with the plots of the SF moment $\tau_n$ of the master. If $R_C < 1$, the slave exhibits the CFs and the SFs. Then, the plots of $T_m$ include lighter parts and lacking parts which correspond to SFs. Next, let us fix the slave parameter at $k_S = 0.73$. In this case, if the slave has no pulse-train input $y_M$, it is chaotic. Fig.6(d) and (e) show a bifurcation diagram of $T_m$ and $R_C$ for $k_M$. Compared to Fig.6(b), the plots of $T_m$ in Fig.6(d) have fewer lacking parts: a chaotic slave may synchronize with more various pulse-train input $y_M$ than a periodic slave. Fig.6(f) shows a bifurcating diagram of $R_C$ for $k_S$ and $k_M$.

## 4. Implementation

We have implemented the network as the following. The base signals $B_M(t)$ and $B_S(t)$ are generated using a function generator. The circuit is implemented by NJM13600(current source), LM339(comparator), 4066(switch), 14071(OR gate) and TL071(op-amp). In order to obtain the firing pulse with appropriately short width, we added monostable multivibrators(4538) at the outputs of the comparators. Fig.7 shows laboratory measurements.

## 5. Conclusions

The master-slave network can exhibit various synchronization phenomena due to the interaction of the SFs and the CFs. It is suggested that a chaotic slave can synchronize with more various input pulse-trains than a periodic slave. Future problems include: 1) detailed classification and analysis of the synchronization phenomena; 2) development of the network to an artificial neural system; and 3) design of a simple implementation circuit.

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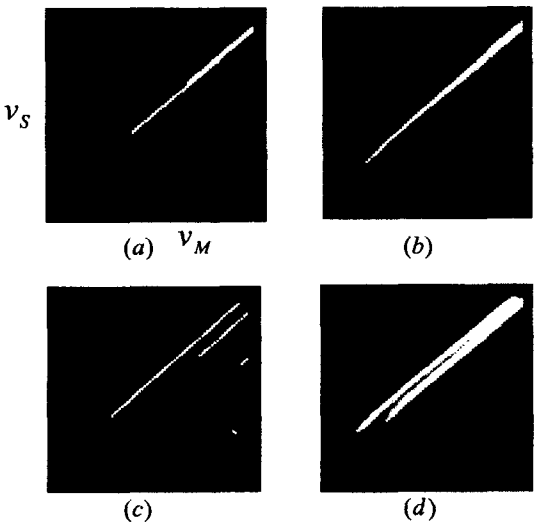


Figure 7. Laboratory measurements. $C \doteq 10$ [nF], $V_F \doteq 2$[V], $V_C \doteq 1.6$[V], $T \doteq 0.5$[ms], $I_M \doteq 32[\mu A]$, $I_S \doteq 29[\mu A]$. $v_M$ and $v_S$: 2V/div. (a)&(b): Periodic slave. $K_S \doteq 0.8$[V]. (c)&(d): Chaotic slave. $K_S \doteq 1.46$[V]. (a) $k_M \doteq 0.8[V]$. (b) $k_M \doteq 1.46[V]$. (c) $k_M \doteq 1[V]$. (d) $k_M \doteq 1.4[V]$. (a)-(d) correspond to Fig.4(a)-(d).