Polynomial Time Solvability of Liveness Problem of Siphon Containing Circuit Nets

Atsushi Ohta and Kohkichi Tsuji
Faulty of Information Science and Technology
Aichi Prefectural University, Nagakute, Aichi, Japan
Tel: +81-561-64-1111, Fax: +81-561-64-1108
E-mail: {ohta,tsuji}@ist.aichi-pu.ac.jp

Abstract: Petri net is an effective modeling tool for concurrent systems. Liveness problem is one of analysis problems in Petri net theory verifying whether the system is free from any local deadlocks. It is well known that computational complexity of liveness problem of general Petri net is deterministic exponential space. Some subclasses, such as marked graph and free choice net, are suggested where liveness problem is verified in less complexity. This paper studies liveness of siphon containing circuit (SCC) net. Liveness condition based on algebraic inequalities is shown. Then polynomial time decidability of liveness of SCC net is derived, if the given net is known to be an SCC net a priori.

1. Introduction

Petri net is an effective modeling tool for concurrent systems[1]. It is a bipartite directed graph consisting of two kinds of nodes, places and transitions. Places and transitions are interpreted as conditions and events, respectively. Places can contain some tokens and the marking, token distribution, represents the state of the Petri net. Marking is changed by firing of a transition, corresponding to the occurrence of event.

Liveness problem is one of analysis problems in Petri net theory. A Petri net is live if every transition is potentially fireable. In other words, a live Petri net exhibits no local deadlocks. It is well known that computational complexity of liveness problem of general Petri net is deterministic exponential space [2].

Some subclasses are suggested where liveness problem is verified in less complexity. For example, liveness problem of marked graph is of deterministic polynomial time complexity. Liveness of bounded free choice system is of the same complexity. Liveness problem of general (unbounded) free choice system is NP-complete [3].

This paper studies liveness of siphon containing circuit (SCC) net. SCC net, is a super class of marked graphs and siphon circuit net. It is shown that an SCC net is live only if it also belongs to the class of trap containing circuit net. This yields an algebraic necessary and sufficient condition for liveness of SCC net. This result shows that liveness problem of SCC net is solvable in polynomial time provided that it is known to be an SCC net beforehand.

2. Definitions

Petri net [1] is a tuple $N = (P, T, F)$, where P and T are disjoint sets of places and transitions, and $F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs. Marking is a mapping from the set of places P to nonnegative integers. A place p has $M(p)$ tokens. Petri net system is a tuple $\Sigma = (N, M_0)$, where M_0 is the initial marking.

The incidence matrix $A = \{a_{ij}\}$ of Petri net N is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } (t_j, p_i) \in F \wedge (p_i, t_j) \notin F \\ -1 & \text{if } (p_i, t_j) \in F \wedge (t_j, p_i) \notin F \\ 0 & \text{otherwise.} \end{cases}$$

Let x be a place or a transition. ${}^\bullet x$ and $x^\bullet$ is defined as ${}^\bullet x = \{y | (y, x) \in F\}$ and $x^\bullet = \{y | (x, y) \in F\}$.

In this paper, Petri net N is assumed to have no isolated nodes. That is, for all $x \in P \cup T$, ${}^\bullet x \cup x^\bullet \neq \emptyset$ holds.

A transition t is fireable if $\forall p \in {}^\bullet t$; $M(p) \geq 1$. This is denoted as $M[t\rangle$. If a fireable transition t fires in M, t removes one token from each place in ${}^\bullet t$ and adds one token to each place in $t^\bullet$. If the resulting marking is M', this is denoted as $M[t\rangle M'$. Let W be a sequence of transitions. If transitions in W can fire in turn from a marking M, this is denoted as $M[W\rangle$ and if the resulting marking M', this is denoted as $M[W\rangle M'$ and M' is said to be reachable from M. The reachability set $R(M)$ is the set of reachable markings from M.

Let $\psi(W)$ be the vector defined as

$$\psi(W)_j = (\text{number of appearance of } t_j \text{ in } W),$$

M' is expressed using M, A, and $x = \psi(W)$ as

$$M' = M + Ax.$$

Let x be a nonnegative $|T|$-vector. Support $\|x\|$ of x is a set of transitions defined as $\|x\| = \{t_j | x_j > 0\}$. Let y be a nonnegative $|P|$-vector. Support $\|y\|$ of y is a set of places defined as $\|y\| = \{p_j | y_j > 0\}$. Let S be a subset of places. Characteristic vector $y(S)$ of S is defined as

$$y(S)_j = \begin{cases} 1 & \text{if } p_j \in S \\ 0 & \text{if } p_j \notin S \end{cases}$$

Let $N = (P, T, F)$ be a Petri net and T_1 be a subset of transitions. $N_1 = (P_1, T_1, F_1)$ is the subnet induced by T_1 if $P_1 = {}^\bullet T_1 \cup T_1{}^\bullet$ and $F_1 = F \cap (P_1 \times T_1 \cup T_1 \times P_1)$. Let $N = (P, T, F)$ be a Petri net and P_1 be a subset of transitions. $N_1 = (P_1, T_1, F_1)$ is the subnet induced by P_1 if $T_1 = {}^\bullet P_1 \cup P_1{}^\bullet$ and $F_1 = F \cap (P_1 \times T_1 \cup T_1 \times P_1)$.

A place p is bounded if there exists a finite integer k such that $M(p) \leq k$ for all reachable marking $M \in$

$R(M_0)$. A Petri net system Σ is bounded if all places are bounded.

A transition t is live if $\forall M \in R(M_0)$; $\exists M' \in R(M)$ *s.t.* $M'[t\rangle$. A Petri net system Σ is live if all transitions are live. A Petri net N is structurally live if there exists a live initial marking.

A sequence of places and transitions

$$u = p_{i_1} t_{i_1} p_{i_2} t_{i_2} p_{i_3} \cdots p_{i_n} t_{i_n} p_{i_1}$$

is a circuit if $p_{i_j} \in {}^\bullet t_{i_j}$ $(j = 1, 2, \ldots, n)$, $t_{i_j} \in {}^\bullet p_{i_{j+1}}$ $(j = 1, 2, \ldots, n-1)$, and $t_{i_n} \in {}^\bullet p_1$. A circuit u is elementary if $p_{i_j} \neq p_{i_k}$ $(j \neq k)$ and $t_{i_j} \neq t_{i_k}$ $(j \neq k)$. Place set of a circuit u is defined as $P_u \equiv \bigcup_{j=1}^{n} \{p_{i_j}\}$. A circuit u is minimal if there exists no circuit v such that $P_v \subset P_u$.

A nonempty set of places S is a siphon if ${}^\bullet S \subseteq S^\bullet$ holds. A siphon S is minimal if no proper subset of S is a siphon. A nonempty set of places S is a trap if ${}^\bullet S \supseteq S^\bullet$ holds.

Property 1: Let S be a siphon. If S has no token in a marking M, then S has no token in any marking reachable from M. Let R be a trap. If R has token(s) in a marking M, then R has at least one token in any marking reachable from M.
(Proof) Straightforward form the definitions of siphon and trap. □

This property implies that if there is a siphon S having no token, then no transition in ${}^\bullet S \cup S^\bullet$ can never fire.

3. Liveness and Incidence Matrix

Two necessary conditions for liveness are reviewed here. Let a and b be vectors of the same size. $a \geq b$ if and only if $a_i \geq b_i$; $\forall i$. $a \gneq b$ if and only if $a \geq b$ and $a \neq b$.

Property 2: Let N be a Petri net and A is the incidence matrix of N. If inequality $A^T y \lneq 0$ has a solution $y \geq 0$, then N has no live marking [1].
(Proof) Let t_j be a transition such that $(A^T y)_j < 0$. For every fireable sequence W such that $M_0[W\rangle M$, $y^T M = y^T(M_0 + A\psi(W)) = y^T M_0 + (A^T y)^T \psi(W) \leq y^T M_0$. Thus $J(M) = y^T M$ is non-increasing. Moreover if $M[t_j\rangle M'$, $J(M') - J(M) = (A^T y)_j < 0$. Since marking M should be nonnegative, $J(M) = y^T M \geq 0$. Thus, t_j can fire at most $\lfloor -(y^T M_0)/(A^T y)_j \rfloor$ times. □

Property 3: Let $\Sigma = (N, M_0)$ be a Petri net and A is the incidence matrix of N. If inequality $y^T M_0 = 0$, $A^T y = 0$ has a solution $y \gneq 0$, then Σ is not live.
(Proof) For every fireable sequence W such that $M_0[W\rangle M$, $y^T M = y^T(M_0 + A\psi(W)) = y^T M_0 + (A^T y)^T \psi(W) = y^T M_0 = 0$. This means no place in $\|y\|$ can have any tokens. Thus all transitions in ${}^\bullet\|y\| \cup \|y\|^\bullet$ can never fire. □

4. SCC net and TCC net

A Petri net Σ is a siphon containing circuit (SCC) net if every elementary circuit of Σ contains a siphon. Equivalently Σ is a SCC net if every minimal circuit is a siphon.

A Petri net Σ is a trap containing circuit (TCC) net if every elementary circuit of Σ contains a siphon. Equivalently Σ is a TCC net if every minimal circuit is a trap.

For example, Petri net N of the figure 1 is an SCC net. This net has five elementary circuits.

$$\begin{aligned} C_1 &= p_1 t_1 p_2 t_5 p_3 t_4 p_1 \\ C_2 &= p_4 t_3 p_5 t_1 p_4 \\ C_3 &= p_4 t_3 p_5 t_2 p_4 \\ C_4 &= p_2 t_2 p_4 t_3 p_5 t_1 p_2 \\ C_5 &= p_1 t_1 p_4 t_3 p_5 t_2 p_3 t_4 p_1 \end{aligned}$$

C_1, C_2 and C_3 are minimal circuit. They are all siphons and traps. C_4 and C_5 are not minimal and they contain a siphon $\{p_4, p_5\}$, which is also a trap. Thus N is an SCC net and is a TCC net.

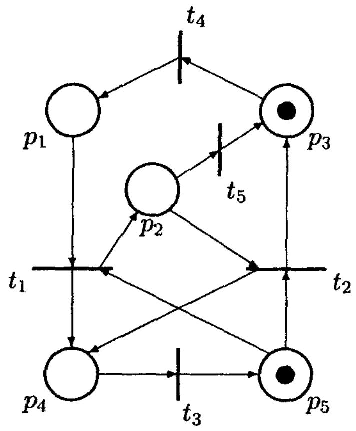

Figure 1. An SCC net.

Here some properties of SCC net and TCC net are reviewed [4], [5], [6], [7].

Property 4: Let N be a SCC(TCC) net and u is a minimal circuit of N.

$$|{}^\bullet t \cap P_u| = |t^\bullet \cap P_u| = 1$$

holds for every transition $t \in {}^\bullet P_u \cap P_u{}^\bullet$ [4].
(Proof) Let $u = p_1 p_2 \cdots p_s p_1$. If there exists a transition t such that $|{}^\bullet t \cap P_u| \geq 2$ and $|t^\bullet \cap P_u| \geq 1$, then u is not minimal. Indeed, if p_i, $p_j \in {}^\bullet t$ and $p_k \in t^\bullet$ (without loss of generality, we can assume $1 \leq i < j \leq k \leq s$) then $u' = p_1 p_2 \cdots p_{i-1} p_i p_k p_{k+1} \cdots p_s p_1$ is a circuit and P'_u is a proper subset of P_u. □

Property 5: Let $N = (P, T, F)$ be a SCC(TCC) net and T_1 is a subset of transition. Then the subnet N_1 induced by T_1 is also a SCC(TCC) net.
(Proof) Proof is done for SCC net, since TCC net is obtained by inverting direction of every arc. Define ${}^\circ Q$ and Q° as ${}^\circ Q = {}^\bullet Q \cap T_1$ and $Q^\circ = Q^\bullet \cap T_1$. N_1 is a SCC net since every circuit C of N_1 is also a circuit of N and C contains a siphon Q in N. Q is also a siphon

in N_1. $^\bullet Q \supseteq Q^\bullet$ implies $^\bullet Q \cap T_1 \supseteq Q^\bullet \cap T_1$ and this means $^\circ Q \supseteq Q^\circ$. □

Property 6: Let N be a TCC(SCC) net and A be the incident matrix of N. Let M_1 and M_2 be two markings of N such that $M_2 = M_1 + Ax$ for some nonnegative integer vector x. If every minimal circuit is marked in M_1 (M_2), then M_2 is reachable from M_1.
(Proof) See reference [5], [6].

Property 7: A TCC net N is structurally live if and only if N has no source place (place without any input transitions). Moreover structurally live TCC net N is live if every minimal circuit is marked [7].
(Proof) (i) N has a source place if and only if $A^T y \leqq 0$ for some $y \geq 0$. Indeed, $A^T y(p_0) \leqq 0$ for any characteristic vector $y(p_0)$ of a source place p_0. Since $A^T y(P_u) \geq 0$ for every minimal circuit u, if $A - Ty \leqq 0$ for some $y \geq 0$, subnet induced by $||y||$ can be assumed to no circuits. This implies N has a source place. (ii) If N has no source place, then inequality $A^T y \leqq 0, y \geq 0$ has no solution. This implies inequality $Ax \geq 0, x > 0$ has an integer solution x. Property 6 implies x is feasible (that is, a firing sequence W such that $\psi(W) = x$ is fireable) from any reachable marking if every minimal circuit is marked in initial marking M_0. Note that every minimal circuit is a trap in TCC net. □

5. Liveness of SCC net

Structural features of live SCC net are shown in the reference [7].

Lemma 1: Live SCC is a TCC net.
(Proof) Assume that a live SCC net Σ is not a TCC net. There exists a minimal circuit u that is not a trap. Let t_0 be a transition in $P_u^\bullet - {}^\bullet P_u$. Property 4 implies that $|{}^\bullet t \cap P_u| = |t^\bullet \cap P_u| = 1$ holds for every transition $t \in {}^\bullet P_u \cap P_u^\bullet$. Thus $y_u^T A \lneqq 0$ holds for characteristic vector y_u of u, since $(y_u^T A)_j = |t^\bullet \cap P_u| - |{}^\bullet t \cap P_u|$. This implies Σ has no live marking (see property 2). □

Lemma 2: Let N be an SCC net that is also a TCC net and A be the incidence matrix of N. $y(P_u)$ satisfies $A^T y(P_u) = 0$ if P_u is the characteristic vector of a minimal circuit u.
(Proof) Every minimal circuit u is a siphon and a trap. Thus ${}^\bullet P_u = P_u^\bullet$ holds. Property 4 implies that for every transition t_j $t_j \notin ({}^\bullet P_u \cup P_u^\bullet)$ or $|{}^\bullet t_j \cap P_u| = |t_j^\bullet \cap P_u| = 1$ holds. In both case $(A^T y(P_u))_j = |t_j^\bullet \cap P_u| - |{}^\bullet t_j \cap P_u| = 0$. □

Based on these properties, liveness condition of SCC net is derived.

Theorem 1: Let Σ be a SCC net. Σ is live if and only if the following conditions are satisfied.
1. Inequality

$$y^T A \lneqq 0, \qquad y \geq 0 \qquad (1)$$

has no solution.
2. Equality

$$y^T A = 0, \qquad y^T M_0 = 0, \qquad y \gneqq 0 \qquad (2)$$

has no solution.

(Proof) (Only if part) In general Petri net, if (1) has a solution, there exists no live marking from property 2. If (2) has a solution Σ is not live from property 3. (If part) Proof of the lemma 1 shows that if an SCC net is not a TCC net, then (1) has a solution. Therefore, if (1) has no solution, Σ is a TCC net. Now assume that (1) has no solution and Σ is a TCC net. Lemma 2 implies that if there exists a minimal circuit having no token in M_0, then (2) has a solution. Thus, if (2) has no solution, every minimal circuit has a token. Σ is live from property 7. □

Theorem 2: Given a Petri net Σ. If it is known a priori that Σ is an SCC net, its liveness is solved in deterministic polynomial time.
(Proof) Both of the conditions are reduced to linear programming problem, that can be solved in deterministic polynomial time. □

Here relation among subclasses is investigated.

Lemma 3: A live and bounded TCC net $\Sigma = (N, M_0)$ is also an SCC net.
(Proof) If a live TCC net Σ is not an SCC net, there exists a minimal circuit u and a transition t such that $t \in {}^\bullet P_u - P_u^\bullet$. Property 4 implies total token count $M(P_u)$ of P_u is nondecreasing since $P_u^\bullet - {}^\bullet P_u = \emptyset$. Moreover $M(P_u)$ increases by one every time t fires. Thus Σ is not bounded. □

Theorem 3: Let LTCC, LBTCC, LSCC, LSCCTCC be classes of live TCC net, live and bounded TCC net, live SCC net, live TCC and SCC net respectively. Then inclusive relation

$$\text{LBTCC} \subset \text{LSCCTCC} = \text{LSCC} \subset \text{LTCC}$$

holds, where $\subset$ means proper subset.
(Proof) Lemma 3 implies the first inclusion. Proper inclusion is shown by the live unbounded SCC net of the figure 2. It is unbounded since firing of $t_4 t_1 t_4 t_1 \cdots$ adds infinitely many tokens in p_4 and p_5 without consuming them. Lemma 1 implies the equality and the last inclusion. Proper inclusion is shown by the live unbounded TCC net of the figure 3. □

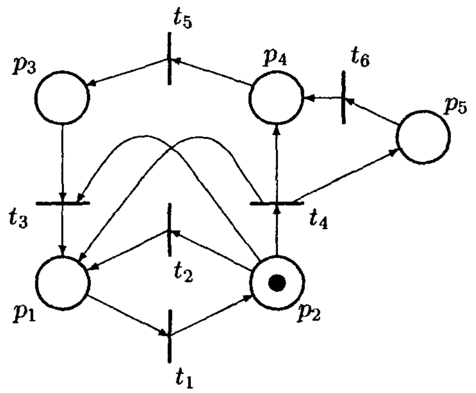

Figure 2. An live unbounded SCC net.

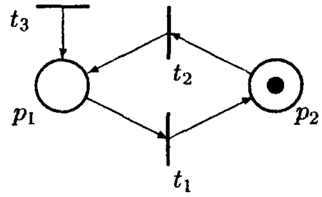

Figure 3. An live unbounded TCC net.

6. Conclusion

Algebraic liveness condition of SCC net is derived. This implies that liveness problem of this subclass is solved in deterministic polynomial time if the give net is assumed to be an SCC net. Future study includes to find computational complexity to decide whether the given net is SCC net.

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