The Fast Lifting Wavelet Transform for Image Coding

Jonghong Shin[1] and Innho Jee[2]
[1]School of Electronics, Electrical and Computer Engineering, Hongik University, Chochiwon, Chungnam, 339-800, KOREA
Tel. +82-41-860-2596, Fax. + 82-41-865-0460
[2]School of Electronics, Electrical and Computer Engineering, Hongik University, Chochiwon, Chungnam, 339-800, KOREA
e-mail : sigs@won.hongik.ac.kr, ijee@wow.hongik.ac.kr

Abstract: We show how any discrete wavelet transform or two band subband filtering with finite filters can be decomposed onto a finite sequence of simple filtering steps, which we call lifting steps but that are also known as ladder structures. We present a self-contained derivations, building the decomposition from the basic principles such as the Euclidean algorithm, with a focus on a applying it to wavelet filtering. This factorization provides an alternative for the lattice factorization, with the advantage that it can also be used in the bi-orthogonal, i.e, non-unitary case. Lifting leads to a speed-up when compared to the standard implementation. We show that this lifting scheme can be applied in image compression efficiently

1. Introduction

In mathematical analysis, wavelets were defined as translates and dilates of one fixed function and were used to both analyze and represent general functions[1][2]. In the mid eighties the introduction of multi-resolution analysis and the fast wavelet transform by Mallat and Meyer provided the connection between subband filters and wavelets[3]. This led to new constructions, such as the smooth orthogonal, and compactly supported wavelets[1]. Later many generalizations to the bi-orthogonal or semi-orthogonal case were introduced. Bi-orthogonality allows the construction of symmetric wavelets and thus linear phase filters. Examples are following: the construction of semi-orthogonal spline wavelets[4], fully bi-orthogonal compactly supported wavelets[2], and recursive filter banks[5]. Several techniques to construct wavelet bases, or to factor existing wavelet filters into basic building blocks are known. Lifting was originally developed to adjust wavelet transform to complex geometries and irregular sampling leading to so-called second generation wavelets. It can also be seen as an alternate implementation of classical, first generation wavelet transforms. The main feature of lifting is that it provides an entirely spatial-domain interpretation of the transform, as opposed to the more traditional frequency-domain based constructions.

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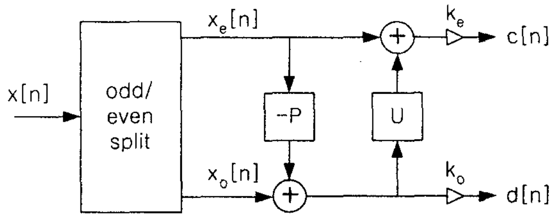

Figure 1 : Typical Lifting Steps : Split, Predict and Update

The local spatial interpretation enables us to adapt the transform not only to the underlying geometry but also to the data, thereby introducing non-linearities while control of the transforms multi-scale properties. A typical lifting is comprised of the steps: Split, Predict, and Update(as shown in Figure1):

Split: Let $x[n]$ be a signal. We first split x[n] into its even and odd poly-phase components $x_e[n] = x[2n]$ and $x_o[n] = x[2n+1]$. If the $x[n]$ corresponds to the samples of an underlying smooth, slowly varying functions, then the even and odd poly-phase components are highly correlated. This correlation structure is typically local, and thus we should be able to accurately predict each odd poly-phase coefficients from the nearly even poly-phase coefficients.

Predict: In the interpolating formulation of lifting, we predict the odd poly-phase coefficients $x_o[n]$ from the neighboring even coefficients $x_e[n]$. The predictor for each $x_o[n]$ is a linear combination of neighboring even coefficients:

$$P(x_e)[n] = \sum_l p_l x_e[n+l] \quad (1)$$

We obtain a new representation of the $x[n]$ by replacing $x_o[n]$ with the prediction residual. This leads to the first lifting step:

$$d[n] = x_o[n] - P(x_e)[n] \quad (2)$$

If the underlying signal is locally smooth, the prediction residual $d[n]$ will small. Furthermore, the new representation contains the same information as the original signal $x[n]$: given the even poly-phase $x_e[n]$ and the prediction residual $d[n]$, we can recover the odd poly-phase coefficients $x_o[n]$ by noting that

$$x_o[n] = d[n] + P(x_e)[n]. \quad (3)$$

This prediction procedure is equivalent to applying a high-pass filter to $x[n]$.

Update: The third lifting step transforms the even poly-phase coefficients $x_e[n]$ into a low-pass filtered and sub-sampled version of $x[n]$. We obtain this coarse approximation by updating $x_e[n]$ with a linear combination of the prediction residuals $d[n]$. We replace $x_e[n]$ with

$$c[n] = x_e[n] + U(d)[n] \quad (4)$$

where $U(d)$ is a linear combination of neighboring d values :

$$U(d)[n] = \sum_l u_l d[n+l] \quad (5)$$

Each lifting steps is always invertible ; no information is lost. Assuming the same P and U are chosen for analysis and synthesis stage, the lifting construction guarantees perfect reconstruction for any P and U. Given $d[n]$ and $c[n]$, we have

$$x_e[n] = c[n] - U(d)[n] \quad (6)$$

and $x_o[n]$ from (3).

The inverse lifting stage is shown in Figure 2. Note that c and d are at half rate, and this transform correspond to a critically sampled perfect reconstruction filter bank. One can show that the update function determines the properties of the dual wavelet and primal scaling function.

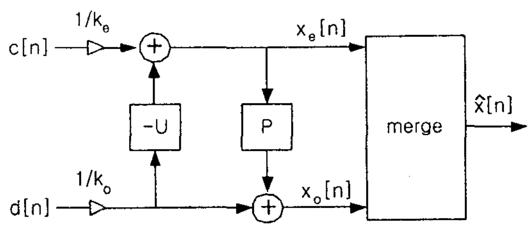

Figure 2 : Typical inverse lifting steps : undo the update, undo the predict, and merge

2. 2-dimensional Wavelet Transform via Lifting

The 2-channel subband for wavelet transform is shown Figure 3.

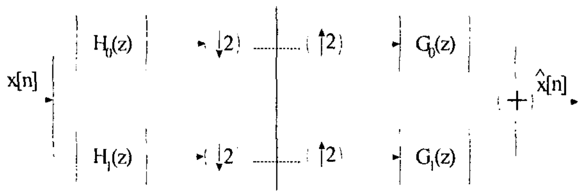

Figure 3 : 2-channel subband coding model

The poly-phase representation for filter is following;

$$h_n(z) = h_{n,e}(z^2) + z^{-1} h_{n,o}(z^2) \qquad (7)$$

where $h_{n,e}$ is even coefficients and $h_{n,o}$ is odd coefficients.

$h_{n,e}(z) = \sum_k h_{n,2k} z^{-k}$ and $h_{n,o}(z) = \sum_k h_{n,2k+1} z^{-k}$

Also, $h_{n,e}(z^2) = \dfrac{h_n(z) + h_n(-z)}{2}$

and $h_{n,o}(z^2) = \dfrac{h_n(z) - h_n(-z)}{2z^{-1}}$

The polyphase matrix is represented as following.

$$P_n(z) = \begin{bmatrix} h_{n,e}(z) & g_{n,e}(z) \\ h_{n,o}(z) & g_{n,o}(z) \end{bmatrix} \qquad (8)$$

The polyphase matrix of synthesis stage is obtained by using similar method. $F(z)$ is defined and if we apply Noble Identity on these filter banks, we can represent 2-channel subband such as Figure 4.

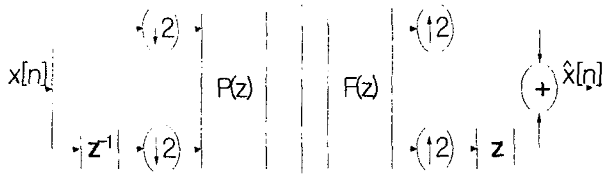

Figure 4 : The polyphase representation for wavelet transform

Lifting theory is the following: The polyphase matrix of wavelet transform in polyphase representation is to be factorized using Euclidean algorithm and to be performed with predict and update stages[6][7]. The Figure 5 is shown the block diagram of predict and update on lifting stage

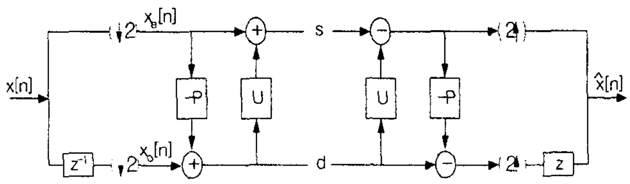

Figure 5 : The block diagram of predict and update on lifting stage

Daubechies 4-tap filter coefficients in Ref[1] is to be factorized into the predict and update stage by using Euclidean algorithm. This factorized polyphase matrix is the following;

$$P(z) = \begin{bmatrix} \frac{\sqrt{3}+1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{3}-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \frac{\sqrt{3}}{4} + \frac{\sqrt{3}-2}{4} z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\sqrt{3} & 1 \end{bmatrix} \qquad (9)$$

The block diagram of analysis 4-tap for wavelet transform via lifting is shown in Figure 6.

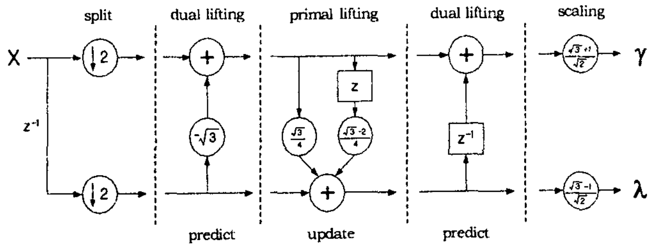

Figure 6 : Orthogonal analysis 4-tap filter for lifting

3. Numerical Simulation and Results

These experimental results of perfect reconstruction of Lenna and Cronkite images on Figure 6 are shown on Figure 7 and 8.

Figure 7 : The perfect reconstruction of Lenna image via 4-tap via lifting scheme

Figure 8: The perfect reconstruction of Cronkite image via 4-tap lifting scheme

The perfect reconstruction PSNR of Lenna image via lifting is 325 dB and PSNR of Cronkite image is 326 dB. Lifting leads to a speed-up when compare to the standard implementation. Also, lifting allows for an in-place implementation of the fast wavelet transform. This means the wavelet transform can be calculated without allocating auxiliary memory.

4. Conclusion

After first optimizing the sub-sampled an up-sampled FIR filters, through the use of some algebra we arrived at a scheme to build a wavelet transform using primal and dual lifting blocks. This decomposition corresponds to a factorization of the poly-phase matrix of the wavelet or sub-band filters into elementary matrices. That such a factorization is possible is well-known to algebraists, it is also used in linear systems theory in the electrical engineering community. We present a self-contained derivations, building the decomposition from the basic principles such as the Euclidean algorithm, with a focus on a applying it to wavelet filtering. This factorization provides an alternative for the lattice factorization, with the advantage that it can also be used in the bi-orthogonal, i.e, non-unitary case. Lifting leads to a speed-up when compared to the standard implementation. We show that this lifting scheme can be applied in image compression efficiently.

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