A High Quality Mesh Generation with Automatic Differentiation for SurfacesDefined by Hamiltonian Lie Algebra

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Abstract: The research on computer graphics (CG) has been actively studied and developed. Namely, many surface/solid models have been proposed in the field of computer aided geometric design as well as the one of CG. Since it is difficult to visualize the complex shape exactly, an approximation by generating a set of meshes is usually used. Therefore it is important to guarantee the quality of the approximation in consideration of the computational cost.

In this paper, a mesh generation algorithm will be proposed for a surface defined by Lie algebra. The proposed algorithm considers the quality in the meaning of validation of invariants obtained by the mesh, using automatic differentiation.

1. Introduction

Recently, since technology of computer graphics (CG) can represent more complicated shape (surface/solid), CG has been used in various engineering fields. Therefore research on surface/solid modeling has been studied, which can represent more complex shape[1].

In the research field, a surface model using linear Lie algebra has been studied and developed, namely in the field of invariant 3-dimensional object recognition and representation[2][3]. The method can represent an object (surface) as small number of parameters, so that it is also expected to be useful in the field of intelligent communication system.

However, since the shape is defined implicitly it is not always approximated with high quality. Therefore, high quality mesh generation method, for defined by liner Lie algebra or Hamiltonian Lie algebra, was proposed[5][6]. However, in the previous method there are some cases of generating not adaptive meshes.

In this paper, in order to generate more adaptively meshes than the previous method in lower computational cost, we shall propose a high quality mesh generation method with automatic differentiation for a surface defined by Hamiltonian Lie algebra.

2. Surface Model by Lie Algebra

2.1 Lie Algebra

Lie group is the C^∞ class differentiable manifold, and normal/tangent vector field on the Lie group is called Lie algebra. Lie group and Lie algebra have the corresponding relation. Then, we can get the shape means global information (Lie group) from normal/tangent vector field means local information (Lie algebra).

2.2 Application of Linear Lie Algebra to Surface Modeling

For a given point $\boldsymbol{p}$ on a surface by linear Lie algebra, its normal vector $\boldsymbol{v}$ is defined as follows:

$$\boldsymbol{v} = A\boldsymbol{p}, \tag{1}$$

where A is a representation matrix such that

$$A = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{21} & 0 \\ 0 & 0 & \lambda_{31} \end{pmatrix} \boldsymbol{P}_{\theta_1}\boldsymbol{Q}_{\phi_1}\boldsymbol{P}_{\psi_1}. \tag{2}$$

Here, $\boldsymbol{P}_{\theta_1}$, $\boldsymbol{Q}_{\phi_1}$ and $\boldsymbol{P}_{\psi_1}$ are rotation matrices with respect to X, Y and Z axis, respectively, and a set of λ_{11}, λ_{21}, λ_{31}, θ_1, ϕ_1 and ψ_1 is called as invariant.

The representation of shapes with Hamiltonian Lie algebra can be represented by extending one with linear Lie algebra to six dimension. The invariant consists of a set of λ_{11}, λ_{12}, λ_{13}, λ_{21}, λ_{22}, λ_{23}, λ_{31}, λ_{32}, λ_{33}, θ_1, ϕ_1, and ψ_1. Then, we calculate A by using Eq.(2). B and C are matrices respectively such that

$$B = \begin{pmatrix} \lambda_{12} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{32} \end{pmatrix},$$

$$C = \begin{pmatrix} \lambda_{13} & 0 & 0 \\ 0 & \lambda_{23} & 0 \\ 0 & 0 & \lambda_{33} \end{pmatrix}. \tag{3}$$

With these matrices, a representation matrix H is defined as

$$H = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \tag{4}$$

and v is obtained as

$$\boldsymbol{v} = H\boldsymbol{p} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}\boldsymbol{p}. \tag{5}$$

Since the surface is implicitly defined as a set of integral curves by Eq.(1), the neighbor point $\boldsymbol{p}_{i+1}$ of $\boldsymbol{p}_i$ is obtained approximately as

$$p_{i+1} = p_i + \Delta t w_{j,i}, \quad i = 0, 1, 2, \cdots \quad j = 0, 1, 2, \cdots \tag{6}$$

where w_i is a tangent unit vector at p_i and Δt is small enough. Since there are not unique tangent vector $w_{j,i}$ at the p_i, in this paper we determine the $w_{j,i}$ as the following under the given $w_{j,0}$:

$$w_{j,i} = w_{j,i-1} - (v_{j,i} w_{j,i-1}) v_{j,i}. \tag{7}$$

3. Mathematical preparation

3.1 Automatic Differentiation

Suppose that f is a function of $x_i, i = 1, \ldots, n$, and that f consists of a set of procedures $w = \phi(u, v)$ or $w = \psi(u)$, where ϕ is a binary operator (e.g., $+, -, \times, \cdots$) and ψ is a unary operator (e.g., $\sin, \cos, \exp, \cdots$). If $\frac{\partial u}{\partial x_i}$ and $\frac{\partial v}{\partial x_i}$ are given, then $\frac{\partial w}{\partial x_i}$ can be obtained with the chain-rule of differentiation. By calculating such $\frac{\partial w}{\partial x_i}$ simultaneously with calculation of the w, $\frac{\partial f}{\partial x_i}$ can be automatically calculated when the value of f is calculated. Primary approximation of f at a certain perturbation h can thus be obtained as follows:

$$f + \frac{\partial f}{\partial x_i} h. \tag{8}$$

Table 1 shows some examples of the rule of $\frac{\partial w}{\partial x_i}$.

Table 1. Examples of the rule $\frac{\partial f}{\partial x_i}$

w	$\frac{\partial w}{\partial x_i}$
$u \pm v$	$\frac{\partial w}{\partial x_i} = \frac{\partial u}{\partial x_i} \pm \frac{\partial v}{\partial x_i}$
$u \times v$	$\frac{\partial w}{\partial x_i} = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}$
u/v	$\left(\frac{\partial u}{\partial x_i} - w \frac{\partial v}{\partial x_i}\right) / v$

This technique is called automatic differentiation. In this paper, calculate primary approximate value of normal vector v_{i+1} at p_{i+1} by using automatic differentiation :

$$v_{i+1} = \left(\frac{\partial v_i}{\partial x}, \frac{\partial v_i}{\partial y}, \frac{\partial v_i}{\partial z}\right) \Delta t w_i + v_i. \tag{9}$$

3.2 Delaunay Triangulation

Delaunay triangulation is a technique which makes a triangle set the convex domain which the point set up on the plane governs. This technique has the feature of not including other points in the circumscribed circle of the obtained triangle. Here we show an outline of delaunay triangulation algorithm as follows:

step1 Set point P and look for a triangle including P.

step2 Look for a triangle with the circumscribed circle including P.

step3 Remove the common side from a set of the triangle discovered at step 1 and step 2, and generate polygon.

step4 Perform triangle division for a polygon at vertices and P.

step5 By repeating step4 from step1, the triangular mesh is generated.

By using this technique, a triangle with little distortion is generable.

4. Algorithm

The previously proposed meshing method[5][6] has a problem that the meshes are too fine around the given initial point, and also rough far from the point. In order to solve the problem, we propose the following meshing method:

Step1 Set a set of invariant, an initial point p_0, an initial direction $w_{0,0}$ of generating point sequence, an initial size Δt of mesh, and a pair of feasible angles β_{min} and β_{max}. Also let $i = j = 0$.

Step2 If $j = 0$ then $w_{j,0} \leftarrow w_{0,0}$. Else, let $w_{j,0}$ be a tangent vector at p_0 on the surface such that a point $p_0 + \Delta t\, w_{j,0}$ is at least Δt far from $p_0 + \Delta t\, w_{k,0}$ where $k = 0, 1, 2, \ldots, j-1$. If there is no $w_{j,0}$ satisfying the condition, finish this algorithm.

Step3 Calculate the normal vector v_i on p_i by Eq.(5), tangent vector w_i by Eq.(7) and neighbor point p_{i+1} by Eq.(6).

Step4 Calculate a normal vector v_{i+1} by Eq.(8) and the angle α between a normal vector v_i and v_{i+1}.

Step5 If $\alpha < \beta_{min}$, then return to **Step2** with $t \leftarrow \delta t$ where $\delta > 1$. If $\alpha > \beta_{max}$ then return to **Step2** with $t \leftarrow \delta' t$ where $\delta' < 1$. Else, p_{i+1} is determined to be a necessary point to generate meshes.

Step6 If the distance of p_{i+1} and the set of points obtained at $j - 1$ is longer than the given threshold, then define a new point between p_{i+1} and the set of points. Then execute this algorithm recursively.

Step7 If p_{i+1} does not satisfy the given terminal condition, return to **Step2** with $i \leftarrow i+1$. Else return to **Step2** with $i \leftarrow 0$ and $j \leftarrow j + 1$.

Step7 Generate meshes by using delaunay triangulation from generated point.

From the above, the high quality meshes can be generated automatically and adaptively, only by setting invariant data, initial point and some paramenters. We note here that the previously proposed method[5][6] cannot generate adaptively meshes around the intial point and far from the point, becouse this method consider the variation only the point on the same integral curve when generate point to do meshes. However in the proposed method in this paper the variation around the given point is considered. Calculate p_{i+1} by approximating a normal vector v_{i+1} using automatic differentiation, and reduce the computational costs.

5. Simulation

We here show a set of results by the proposed algorithm.

Fig1., 2., 4., and 5. show a set of images generated by the previously proposed method. Fig 3. and 6. show a set of images generated by the proposed methpd. All figures show meshes with wireframe. Table 2 and 3 show invariant of surface and number of generated meshes for each surface and computational costs under the 443MHz Ultra SPARC IIi system, respectively.

In Fig1. and Fig2., number of integral curves from the initial point to generate meshes are invariable. In Fig1., the number is 15 while in Fig 2., the number is 40. In Fig3., define the number by the variation the given point.

Compared Fig1. and Fig2. with Fig3., in Fig1., the meshes are rough far from the initial point, and in Fig2., the meshes are too fine around the point, in Fig.3 the meshes are adaptive anywhere. From the Fig.2., 3. and Table 3, it is seen that defining adaptively the number of integral curves and approximating a normal vector using automatic differentiation are effective for reducing computational cost with guaranteeing the quality.

In Fig4. ~ Fig 6., it is a nonlinear case. In the Fig.4 ~ Fig 6., it is the same what it has stated above.

Table 2. Invariant of figures

	invariant					
Fig.1, ~, Fig.3		1	1	1	0	0
		0	0	0	0	0
		0	0	0	0	0
Fig.4, ~, Fig.6	-0.1	-0.5	-0.6	-0.4	1.0	-1.0
	-0.5	0.6	0.4	0.1	0.5	0
	0.1	0.2	-0.4	0	0.5	0.5

Table 3. Parameter of figures

	number of meshes	computational costs [s]
Fig.1	945	2.47
Fig.2	4360	4.51
Fig.3	3605	3.54
Fig.4	709	2.64
Fig.5	2532	4.41
Fig.6	1497	4.00

6. Conclusion

In this paper, we have proposed an adaptive meshing algorithm with the guaranteed quality for the surface defined by Hamiltonlian Lie algebra.

In the proposed method, the size of mesh is determined adaptively and automatically in consideration of the number of integral curve from the initial point to generate meshes. The computational costs is reduced by approximating a normal vector using automatic differentiation, and reduce the computational costs. Furthermore, in consideration of the proposed termination condition, the method can be applied to closed surfaces.

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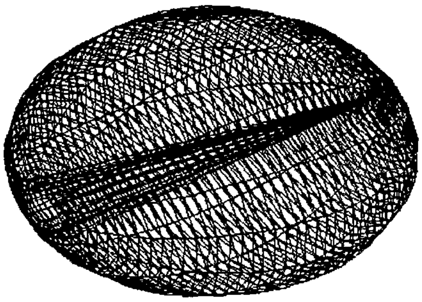

Figure 1. number of integral curve ; 15

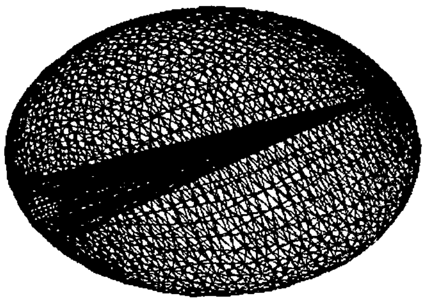

Figure 2. number of integral curve ; 40

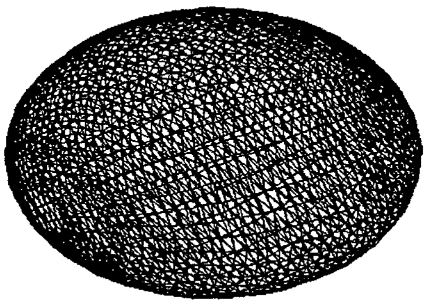

Figure 3. proposed method

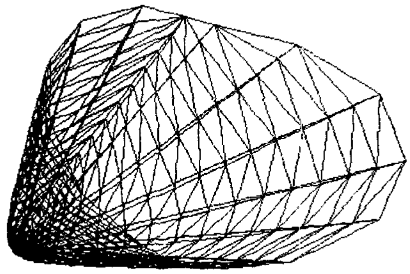

Figure 4. number of integral curve ; 15

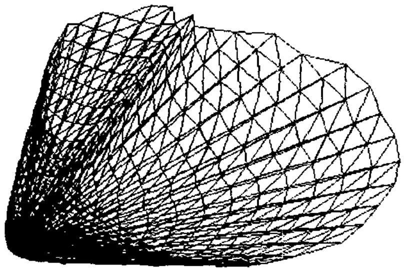

Figure 5. number of integral curve ; 40

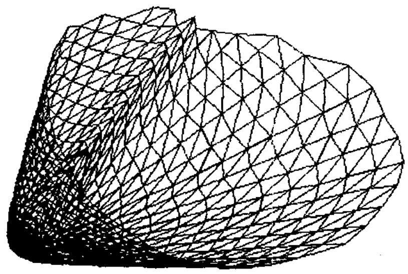

Figure 6. proposed method