A High Quality Mesh Generation for Surfaces in the Use of Interval Arithmetic

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Abstract: In this paper, a high quality mesh generation method by using interval arithmetic is proposed. In the proposed method, the variance of a tangent vector at the point is considered by the automatic differentiation. From the variance, sampling points on the surface are judged whether it is adequate or not, which is calculated by the interval arithmetic. Then Delaunay triangulation is performed to the obtained sampling points, and a set of meshes is generated. The proposed method is hard to overlook the local variation of surfaces.

1. Introduction

Since technology of computer graphics(CG) can represent more complicated shape(surface/solid), CG has been used in various engineering field. Therefore research on surface/solid modeling has been studied, which can represent more complex shape.

However in general, it is difficult to represent exactly the complicated shape, because it takes huge computational cost in visualization. Therefore, an approximation by a set of meshes is often used. It is obviously seen that the quality of approximation depends on suitable size of meshes for the shape of the object. Therefore, in order to approximate surfaces, it is necessary to consider the variation of surfaces[1]. Which guarantees accuracy of the generated meshes according to shape of given surface in suitable computational cost.

In this paper, a high quality mesh generation method by using interval arithmetic[2] is proposed. In this method, the variance of a tangent vector at the point is considered by the automatic differentiation[2][3]. From the variance, sampling points on the surface are judged whether it is adequate or not, which is calculated by the interval arithmetic. Then Delaunay triangulation[4][5] is performed to the obtained sampling points, and a set of meshes is generated. The proposed method is hard to overlook the local variation of surfaces in the use of interval arithmetic.

The method is useful for visualization and computer aided geometric design(CAGD) as well as CG.

2. Parametric Surface

When we define shape of a solid for the computer graphics, a set of parametric surfaces is usually used especially in the related area in CAGD[6][7][8]:

$$P(u,v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} P_x(u,v) \\ P_y(u,v) \\ P_z(u,v) \end{pmatrix}, \qquad (1)$$

where $u_{min} \le u \le u_{max}$ and $v_{min} \le v \le v_{max}$. In consideration of smoothness of the shape, $P(u,v)$ is at least bi-cubic algebra.

2.1 Bézier Surface

Bézier surface is defined form only by the position vector $Q_{i,j}$ called control point. The expression form of $m \times n$ Bézier surface is as follows.

$$P(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{i,m}(u)B_{j,n}(v)Q_{i,j} \qquad (2)$$

where m and n are degree, and $0 \le u, v \le 1$. $B_{i,m}(u)$ and $B_{j,n}(v)$ are Bernstein basis function as follows.

$$B_{i,m}(u) = \frac{m!}{(i! \times (m-i)!)}(1-u)^{m-i}u^i \qquad (3)$$

Bézier surface has the form which smoothes a polygon which connected between control points. Therefore, it is excellent in respect of form control. However, if we will connect surfaces smoothly, some restricted conditions are necessary.

2.2 B-spline Surface

As for Bézier surface, movement of a control point will affect the whole surface. B-spline surface conquered this fault. The expression form of $m \times n$ B-spline surface is as follows.

$$P(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{i,K}(u)B_{j,K}(v)Q_{i,j} \qquad (4)$$

where K is order. $B_i(u), B_j(v)$ are as follows.

$$B_{i,1}(u) = \begin{cases} 1 & (t_i \le u < t_{i+1}) \\ 0 & (u < t_i, t_{i+1} \le u) \end{cases}$$

$$B_{i,K}(u) = \frac{u - t_i}{t_{i+K-1} - t_i} B_{i,K-1}(u) + \frac{t_{i+K} - u}{t_{i+K} - t_{i+1}} B_{i+1,K-1}(u) \quad (K \geq 2) \tag{5}$$

where t_i is knot which must be monotonically increasing.

B-spline surface has the feature that one surface can express two or more segments. Since it is in agreement to a differentiation vector between segments, B-spline surface is excellent in respect of connection of surfaces.

2.3 Rational B-spline Surface

By giving weight to the control point of B-spline surface, it is rational B-spline surface which was rationalized. The expression form of $m \times n$ rational B-spline surface is as follows.

$$P(u,v) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} B_{i,K}(u) B_{j,K}(v) w_{i,j} Q_{i,j}}{\sum_{i=0}^{m} \sum_{j=0}^{n} B_{i,K}(u) B_{j,K}(v) w_{i,j}} \tag{6}$$

where $w_{i,j}$ is weight. $B_{i,K}(u), B_{j,K}(v)$ are the same as B-spline surface's ones.

Rational B-spline surface has the feature of B-spline surface, and can express the surface which took the weight of a control point into consideration.

3. Mathematical Preparation

3.1 Interval Arithmetic

Suppose that a certain variable has an interval of real number $[x_1, x_2](x_1 \leq x_2)$ as a value. Interval arithmetic is operation which considers a range $\{\varphi(a,b) | a \in [a_1, a_2], b \in [b_1, b_2]\}$ or $\{\phi(a) | a \in [a_1, a_2]\}$ as a result, when the interval $[a_1, a_2]$ and $[b_1, b_2]$ are given to the operands of a four arithmetic operations $\varphi(a, b)$ (Eq. (7)) or an elementary function $\phi(a)$.

$$\begin{aligned}
[a_1, a_2] + [b_1, b_2] &= [a_1 + b_1, a_2 + b_2] \\
[a_1, a_2] - [b_1, b_2] &= [a_1 - b_2, a_2 - b_1] \\
[a_1, a_2] \times [b_1, b_2] &= [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \\
&\quad \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)] \\
[a_1, a_2] / [b_1, b_2] &= [\min(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2), \\
&\quad \max(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)] \\
&\quad (0 \notin [b_1, b_2])
\end{aligned} \tag{7}$$

It is seen that the interval arithmetic can include the exact value in the interval. In that meaning, its accuracy is validated. However in practice, it is no meaning when the interval is wide.

3.2 Automatic Differentiation

Suppose that f is a function of $x_i, i = 1, \ldots, n$, and that f consists of a set of procedures $w = \phi(u, v)$ or $w = \psi(u)$, where ϕ is a binary operator (e.g., $+, -, \times, \cdots$) and ψ is a unary operator (e.g., sin, cos, exp, $\cdots$). If $\frac{\partial u}{\partial x_i}$ and $\frac{\partial v}{\partial x_i}$ are given, then $\frac{\partial w}{\partial x_i}$ can be obtained with the chain-rule of differentiation. By calculating such $\frac{\partial w}{\partial x_i}$ simultaneously with calculation of the w, $\frac{\partial f}{\partial x_i}$ can be automatically calculated when the value of f is calculated. Primary approximation of f at a certain perturbation h can thus be obtained as follows:

$$f + \frac{\partial f}{\partial x_i} h. \tag{8}$$

Table 1 shows some examples of the rule of $\frac{\partial w}{\partial x_i}$.

Table 1. Examples of the rule $\frac{\partial f}{\partial x_i}$

w	$\frac{\partial w}{\partial x_i}$
$u \pm v$	$\frac{\partial w}{\partial x_i} = \frac{\partial u}{\partial x_i} \pm \frac{\partial v}{\partial x_i}$
$u \times v$	$\frac{\partial w}{\partial x_i} = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}$
u/v	$\left(\frac{\partial u}{\partial x_i} - w \frac{\partial v}{\partial x_i}\right) / v$

This technique is called automatic differentiation. Unlike numerical differentiation, automatic differentiation does not need the information on the neighbor of the point. In addition, the technique can be used for an algorithm including iteration or branch as well as some formulas. In this paper, the features is applied to gain the variation of surfaces.

3.3 Delaunay Triangulation

Delaunay triangulation is a technique which makes a triangle set the convex domain which the point set up on the plane governs. This technique has the feature of not including other points in the circumscribed circle of the obtained triangle. Here we show an outline of Delaunay triangulation algorithm as follows:

step1 Set point P and look for a triangle including P.

step2 Look for a triangle with the circumscribed circle including P.

step3 Remove the common side from a set of the triangle discovered at step 1 and step 2, and generate polygon.

step4 Perform triangle division for a polygon at vertices and P.

step5 By repeating step4 from step1, a triangular mesh is generated.

By using this technique, a triangle with little distortion is generable.

4. Proposed Point Group Generation for Surfaces

Here we show an outline of the proposed algorithm as follows:

step1 Set control points and knot vector (only in the case of rational B-spline curve[7]), default sampling width Δt between the neighbor points and threshold β. Also set surface parameter u,v be 0, respectively. Then define a set of sampling points which are vertices of initial meshes, and are located in grid with width Δt on u-v plane.

tep2 For a point in the point set, calculate the point P on the surface, the first order derivative of each direction u,v, and the unit normal vector n at P.

tep3 Calculate the unit normal vector n_u,n_v on the point which is Δt away from P in each of u,v direction. Then, interval normal vector N_u is obtained by n and n_u. Calculate N_v as the same as N_u(see Figure 1).

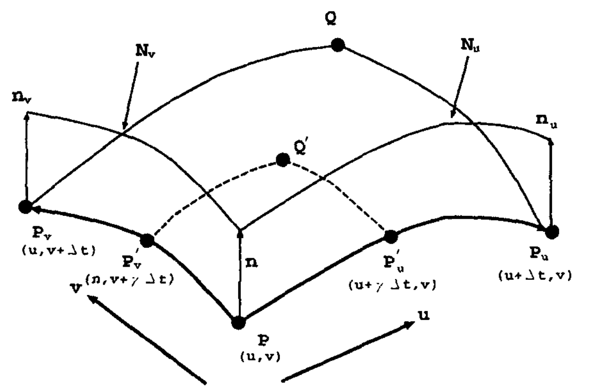

Figure 1. Interval normal vector

tep4 Calculate the inner product interval α by using interval arithmetic between N_u and N_v. If the center of α is smaller than β, since Δt is too large, add the point P'_u, P'_v, Q' to the point set defined in step1, where they are the vertex of the square of one side $\gamma\Delta t (0 < \gamma < 1)$ including P which is one vertex. Then return to step3 with $\Delta t \leftarrow \gamma\Delta t$. If $\alpha \geq \beta$, return to step 2 for another point in the set.

By the conventional method, sampling points are generated from the variation of surfaces only at eight neighborhood points separately. So, it is possible that redundant points are generated or the variation of surfaces is overlooked at the portion which changes from a loose surface to a steep surface. But in the proposed method, by the interval arithmetic we can consider a range as a result. Besides, the variation to each of u, v direction is considered together. Therefore, it is hard to overlook the local variation of surfaces.

5. Simulation

Here, we show some results by the proposed algorithm. Figures 2, 3, and 4 show images when the proposed method is applied to Bézier, B-spline and rational B-spline surface, respectively. In these images, the segments connect sampling points which are arranged at equal interval. From these images we can see the shape of the surfaces easily. Namely, the area where the curvature is large consists of a lot of points, and the area where the curvature is small consists of a few ones. Figure 5, 6, and 7 show images when Delaunay triangulation is applied to the point group which is generated. It is seen that the proposed method can generate a set of mesh efficiently which express the shape of the surfaces. In Figure 2, 3 and 4, sampling points according to the variation of surfaces are generated at the portion which changes from a loose surface to a steep surface.

6. Conclusion

In this paper, a high quality mesh generation method by using interval arithmetic for surfaces is proposed. By the proposed method, it is hard to overlook the local variation of surfaces, because the interval arithmetic recognizes the variation of surface in domain unit.

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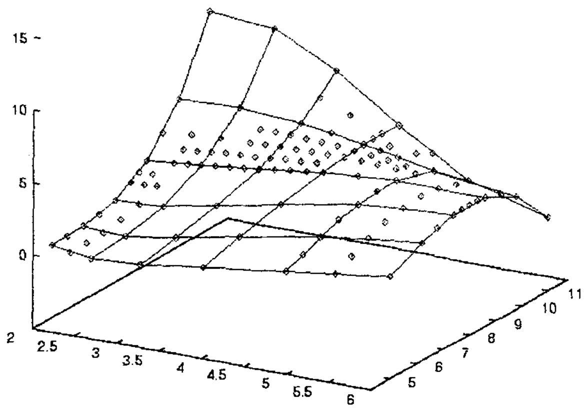

Figure 2. Obtained points on Bézier surface by the proposed method

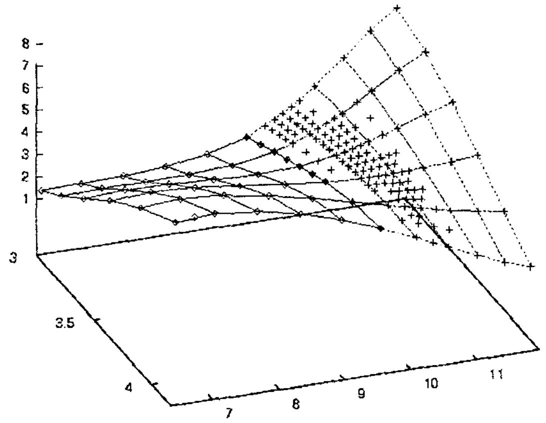

Figure 3. Obtained points on B-spline surface by the proposed method

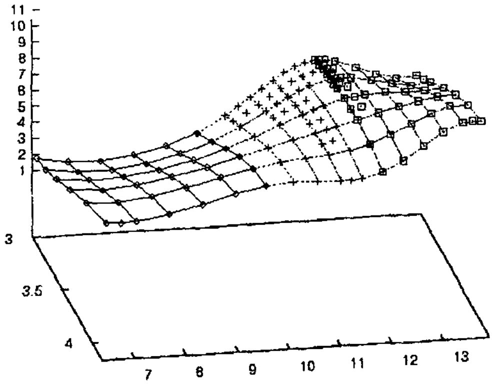

Figure 4. Obtained points on rational B-spline surface by the proposed method

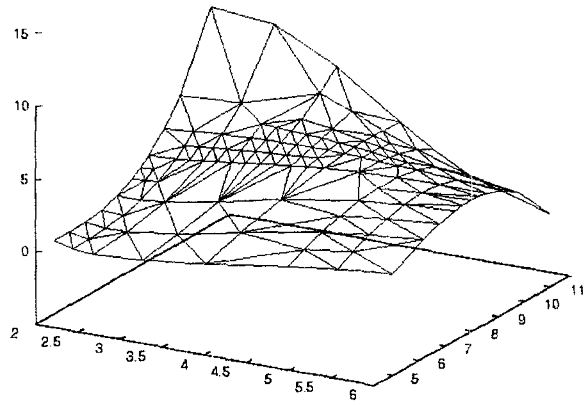

Figure 5. Delaunay triangulation for the obtained points on Bézier surface by the proposed method

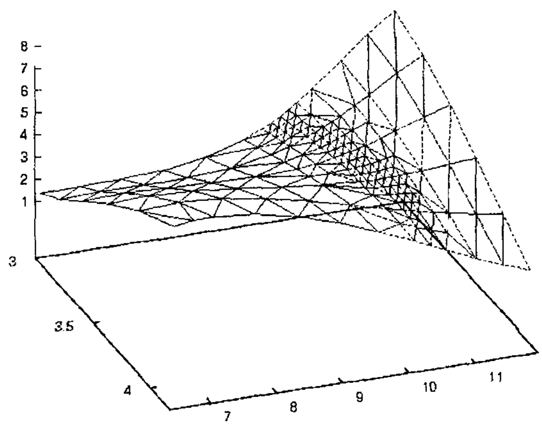

Figure 6. Delaunay triangulation for the obtained points on B-spline surface by the proposed method

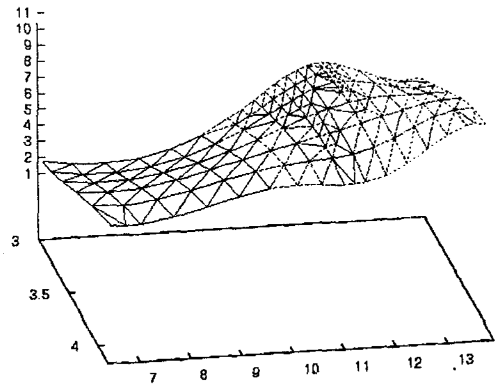

Figure 7. Delaunay triangulation for the obtained points on rational B-spline surface by the proposed method