Characterization of some classes of distributions related to operator semi-stable distributions.

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ABSTRACT

For a positive integer m, operator m-semi-stability and the strict operator m-semi-stability of probability measures on R^d are defined. The operator m-semi-stability is a generalization of the definition of operator semi- stability with exponent Q. Translation of strictly operator m-semi-stable distribution is discussed.

Keywords: operator semi-stability, semi-stability, operator stability, strictly semi-stability

1. Introduction

Let m be a positive integer. In [3], the classes of m-semi-stable and strictly m-semi-stable distributions on R^d were studied. In one dimension, they are first investigated by Lévy [4]. The characterization of these classes on R was developed by Linnik [5], Shimizu [9], Ramachandran and Rao [6], and others. Extension to multidimension was done by Krapavickaitė(1980) and Choi [3]. Here we extend those classes to linear operator cases.

Let $I(R^d)$ be the collection of infinitly divisible distributions on R^d. The characteristic function of $\mu \in I(R^d)$ is denoted by $\widehat{\mu}(z)$, $z \in R^d$. Let $M_+(R^d)$ be the class of linear operators on R^d all of whose eigenvalues have positive real parts. Let $0 < b_l < 1$, $Q \in M_+(R^d)$ and m a positive integer in this paper throughout. We call a distribution μ on R^d operator m-semi-stable if $\mu \in I(R^d)$ and there exist real numbers b_l, c_l, $l = 1, 2, \cdots, m$, and a vector $\gamma \in R^d$ satisfying

$$c_l > 0, \quad \sum_{l=1}^{m} c_l > 1, \quad \text{and} \quad \sum_{l=1}^{m} b_l c_l = 1$$

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such that

$$\widehat{\mu}(z) = e^{\langle \gamma, z\rangle} \Pi_{l=1}^{m} \widehat{\mu}(b_l^{Q'} z)^{c_l}. \tag{1.1}$$

The class of distributions satisfying (1,1) is denoted by $OSS(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$. We call distributions in $OSS(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$ $(Q, b_1, \cdots, b_m, c_1, \cdots, c_m)$-*semi − stable*. Here $\langle , \rangle$ is the Euclidean inner product in R^d and Q' is the adjoint of Q.

Further, a distribution μ on R^d is strictly operator m-semi-stable if $\mu \in OSS(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$ satisfying

$$\widehat{\mu}(z) = \Pi_{l=1}^{m} \widehat{\mu}(b_l^{Q'} z)^{c_l}. \tag{1.2}$$

The class of distributions satisfying (1.2) is denoted by $OSS_0(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$. We call distributions in $OSS_0(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$ strictly $(Q, b_1, \cdots, b_m, c_1, \cdots, c_m)$-*semi − stable*. The distribution satisfying (1.1) with $Q = I$ is a m-semi-stable distribution in the sense of [3].

The main purpose of this paper is to obtain a characterization of translations of strictly operator $(Q, b_1, \cdots, b_m, c_1, \cdots, c_m)$-semi-stable distributions and to discuss relations between translation of strictly operator semi-stable distribution and translation of strictly operator $(Q, b_1, \cdots, b_m, c_1, \cdots, c_m)$-semi-stable distribution.

Our results in this paper are extension of results in [1,2,3], [7] and [8].

2. Preliminaries

We begin with some notation. Let θ_j, $1 \le j \le q + 2r$ denote all distinct eigenvalues of Q such that $\theta_1, \ldots, \theta_q$, are real if $q \ge 1$ and that $\theta_{q+1}, \ldots, \theta_{q+2r}$, are non-real and $\theta_j = \overline{\theta_{j+r}}$ for $q + 1 \le j \le q + r$ if $r \ge 1$. Let $\theta_j = \alpha_j + i\beta_j$ where α_j and β_j are real numbers. Let $f(\zeta)$ be the minimal polynomial of Q with $f(\zeta) = f_1(\zeta)^{n_1} \ldots f_{q+r}(\zeta)^{n_{q+r}}$, where $f_j(\zeta) = \zeta - \alpha_j$ for $1 \le j \le q$ and $(\zeta - \alpha_j)^2 + {\beta_j}^2$ for $q + 1 \le j \le q + r$. We write W_j for the kernel of $f_j(Q)^{n_j}$ in R^d, $1 \le j \le q + r$. We denote the kernel of $(Q - \theta_j)^{n_j}$ in C^d, $1 \le j \le q + 2r$, by V_j. Let T_j be the projector onto V_j. We denote

$$D_j = \{(Q - \theta_j)v : v \in V_j\} \quad \text{in} \quad C^d, \quad 1 \le j \le q + 2r.$$

Let P_j be the projector onto D_j in C^d, $1 \le j \le q + 2r$.

We easily show the following proposition.

Proposition 2. 1 *Suppose that* 1 *is not an eigenvalue of* $\sum_{l=1}^{m} c_l b_l^{Q}$. *Then any* $\mu \in OSS(b_1, \cdots, b_m, c_1 \cdots, c_m, Q)$ *is a translation of a strictly* $(Q, b_1, \cdots, b_m, c_1 \cdots, c_m)$-*semi-stable distribution.*

We set $J = \{j : 1 \leq j \leq q+2r \quad \text{satisfying} \quad \sum_{l=1}^{m} c_l b_l^{0_j} = 1 \quad \text{and} \quad \alpha_j > 1/2\}$ and $\Gamma = \{j : 1 \leq j \leq q+r \quad \text{satisfying} \quad \alpha_j > 1/2\}$. Let $W_\Gamma = \oplus_{j\in\Gamma} W_j$, and let

$$S_\Gamma = \{\xi \in W_\Gamma : \mid \xi \mid = 1, \mid u^Q \xi \mid > 1 \quad \text{for all} \quad u > 1\}.$$

Any $x \in W_\Gamma$ is uniquely expressed as $x = u^Q \xi$ with $\xi \in S_\Gamma$ and $u \in (0, \infty)$.

For some $0 < b < 1$, let $OSS(b, Q)$ be the class of $\mu \in I(R^d)$ such that

$$\widehat{\mu}(z) = e^{\langle \gamma, z \rangle} \widehat{\mu}(b^{Q'} z)^c$$

for some $c > 0$ and $\gamma \in R^d$. Distributions in $OSS(b, Q)$ are called *(Q, b)-semi-stable.* For some $0 < b < 1$, $\mu \in OSS_0(Q, b)$ means that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)^c$$

for some $c > 0$. Distributions in $OSS_0(b, Q)$ are called *strictly- (Q, b)-semi-stable.* We note that operator 1-semi-stable distribution is (Q, b)-semi-stable distribution. For any $\rho > 0$, $A_m(0)$ and $A_m(\rho)$ are, respectivly, the sets of all m-tuples $(b_1, \cdots, b_m)$ with $0 < b_j < 1$, $j = 1, \cdots, m$, satisfying the following conditions.

$A_m(0)$: for some l and i, $logb_l/logb_i$ is an irrational number,

$A_m(\rho)$: $logb_l/logb_i$ is a rational numbers for every l and i, and $m_l = -logb_l/\rho$, $l = 1, \cdots, m$, are positive integers with their greatest common factor equal to one. Using tool in Lemma 2.1 in [1], we can show the following lemma.

Lemma 2. 2 *For $1 \leq j \leq q + 2r$, set*

$$g_{j,0}(b_1 \cdots, b_m, \xi) = \int_0^\infty u^{0_j} T_j \xi \sum_{l=1}^{m} c_l \left(\frac{1}{1+|u^Q \xi|^2} - \frac{1}{1+|(\frac{u}{b_l})^Q \xi|^2} \right) d\left(\frac{-H_\xi(u)}{u} \right),$$

where $H_\xi(u)$ will be given in Section 3. Then the function $g_{j,0}(b_1 \cdots, b_m, \xi)$ is well-defined, bounded, and measurable on S_Γ.

3. Main Results

Any $\mu \in I(R^d)$ has the Lévy representation (A, ν, γ), which means

$$\widehat{\mu}(z) = exp\left[i\langle \gamma, z \rangle - \frac{1}{2}\langle Az, z \rangle + \int_{R^d} G(z, x)\nu(dx) \right],$$

with $G(z, x) = e^{i\langle z,x \rangle} - 1 - i\langle z, x \rangle(1 + |x|^2)^{-1}$. Here $\gamma \in R^d$, A(called the Gaussian covariance of μ) is a symmetric nonnegative-definite operator on R^d, and ν (called the Lévy measure of μ) is a Lévy measure satisfying $\nu(\{0\}) = 0$ and

$$\int_{R^d - \{0\}} |x|^2(1 + |x|^2)^{-1}\nu(dx) < \infty.$$

These A, ν, and γ are uniquely determined by μ. When $\nu = 0$, we call μ a Gaussian distribution. If $A = 0$, then we call μ a purely non-Gaussian distribution.

The following Theorem 3.1 characterizes the class of all purely non-Gaussian operator m-semi-stable distributions. But we do not treat the whole structure of Gaussian operator m-semi-stable distributions. The complete description of Gaussian operator stable distributions and Gaussian operator semi-stable distributions is respectively obtained by Sato[7,8] and Choi [2].

Theorem 3. 1 *Let μ be a $(Q, b_1, \cdots, b_m, c_1, \cdots, c_m)$-semi-stable distribution on R^d with Lévy representation $(0, \nu, \gamma)$. Then, $\mu \in OSS(b_1, \cdots, b_m, c_1, \cdots, c_m, Q)$ if and only if*

$$\nu(E) = \int_{S_\Gamma} \lambda(d\xi) \int_0^\infty I_E(u^Q\xi) d\left(-H_\xi(u)u^{-1}\right), \quad E \in \mathcal{B}(R^d),$$

where

(i) λ is a finite measure on S_Γ,

(ii) $H_\xi(u)$ is a real-valued function being right-continuous in $u \in (0, \infty)$ and measurable in $\xi \in S_\Gamma$ such that $H_\xi(u)u^{-1}$ is decreasing(in the wide sense allowing flatness), $H_\xi(1) = 1$, $H_\xi(bu) = H_\xi(u)$ for any u and ξ and in addition, one of the following (a) and (b):

(a) $(b_1, \cdots b_m) \in A_m(0)$, $H_\xi(u) = 1$,

(b) $(b_1, \cdots b_m) \in A_m(\rho)$, $H_\xi(bu) = H_\xi(u)$.

Theorem 3. 2 *Let μ be as in Theorem 3.1. Then μ is a translation of a strictly $(Q, b_1, \cdots, b_m, c_1, \cdots, c_m)$-semi-stable distribution if and only if*

$$\int_{S_\Gamma} (I - P_j) g_{j,0}(b_1, \cdots b_m, \xi) T_j \xi \lambda(d\xi) = 0 \quad for \quad j \in J.$$

Theorem 3. 3 *Let $OSS(b_1, \cdots, b_m, c_1, \cdots, c_m, Q) = OSS(b, Q)$ for some b. If $\mu \in OSS_0(b_1, b_2, \cdots, b_m, c_1, \cdots, c_m, Q)$, then μ is a translation of a strictly (Q, b)-semi-stable distribution.*

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