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서버 유휴 시의 고객 집단 도착과 서버 다운이 존재하는 M/G/1 모형의 분석

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M/G/1 queue with disasters and mass arrival when empty

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Recently there has been an increasing interest in queueing models with disasters. Upon arrival of a disaster, all the customers present are flushed out. Queueing models with disasters have been applied to the problems of failure recovery in many computer networks systems, database systems and telecommunication networks. In this paper, we suggest the steady state and sojourn time distributions of the M/G/1 model with disaster and mass arrival when the system is empty.

Keywords : Disaster, M/G/1 model, Steady-State Distribution, Sojourn Time Distribution

1. Introduction

Queueing models with disasters have been applied to the problems of failure recovery in many computer network systems, database systems and telecommunication networks. In computer systems, sudden failures such as power-off or server breakdown may be considered as disasters. When the computer system fails, all the ongoing jobs are turned useless and thus cancelled. Queueing system with disasters can be considered as the basic model for computer systems with potential virus working. In telecommunication networks, breaking down of a node can be considered as a disaster. When such situation occurs, messages in the queue (node) must take another route. Calamities, such as damages to road by earthquake and catastrophes in the pest control problem (Kyriakidis and Abakuks [2]), may also be considered as disasters.

In this paper, we assume the $M/G/1$ model with disasters, in which a random number (k) customers immigrate into the system as a batch when the system is empty. During the service periods, customers enter the system individually. This model is a generalization of the model of Chen and Renshaw [1] which assumes exponential service times (whereas ours assumes general service times). In this model, when the queue is empty, a supervisor moves a whole batch of k customers at rate α_k

into the queue. In a situation of light traffic, this can greatly increase the mean inter-idle time of the server, and hence his working efficiency (see Chen and Renshaw [1]). In addition, there are disasters who make all the customers in the system be simultaneously removed at rate $\delta > 0$. For example, a supervisor concerned about a stressed-out server could remove his entire workload at rate δ, allow him to take an exponential $(1/\alpha_0)$ break and then start him afresh with *k* new customers with probability α_k / α_0.

The purpose of this paper is to obtain the steady state distribution and the sojourn time distribution. To obtain the results, we need to define α_k $(k = 1,2,\cdots)$ as the rate at which a whole batch of *k* customers arrives into the queue when the system is empty. Then,

$$\alpha_0 = \sum_{k=1}^{\infty} \alpha_k .$$

Additionally, for the convenience of calculation, we define

$$\alpha(z) = \sum_{k=1}^{\infty} \alpha_k \cdot z^k ,$$

and

$$\alpha'(1) = \frac{d}{dz}\alpha(z)|_{z=1} .$$

As we mentioned, customers enter the system in a group when the system is empty. We define P_g as the probability that an arbitrary customer enters the system in a group. Let us further define $\bar{\pi}_n$ as the probability that there are *n* customers at a test customer's arrival epoch and $\bar{\Pi}^*(z,\theta)$ as the joint transform of the number of customers and the residual service time at a test customer's arrival epoch (alike with $P^*(z,\theta)$ in section 2).

2. Steady-State Distribution

We use the supplementary variable technique to obtain the steady state distribution for the system size N. In this model, we assume the general service time with the density function of $s(x)$ and Laplace transform $S^*(\theta)$. As supplementary variables, we use the remaining service time S_R. Denoted by

$$P_n(x)dx = P\{N = n, x < S_r \le x + dx\}$$

and

$$P_n = \int_0^{\infty} P_n(x)dx .$$

Using these probabilities, we get the steady-state equations as

$$P_0(t+dt) = P_0(t)(1-\alpha_0 dt) + P_1(0,t)dt + \delta\sum_{n=1}^{\infty}\int_0^{\infty} P_n(y,t)dtdy$$

$$P_1(x-dt,t+dt) = P_1(x,t)(1-\lambda dt - \delta dt) + P_0(t)\alpha_1 s(x)dt + P_2(0,t)s(x)dt$$

$$P_n(x-dt,t+dt) = P_n(x,t)(1-\lambda dt - \delta dt) + P_0(t)\alpha_n s(x)dt + P_{n-1}(x,t)\lambda dt + P_{n+1}(0,t)s(x)dt \quad (n \ge 2). \qquad (1)$$

By pursuing the steps used in earlier literatures, we obtain the following results.

$$P(z,0) = \sum_{n=1}^{\infty} p_n(0)z^n = \frac{zS^*(\lambda+\delta-\lambda z)}{S^*(\lambda+\delta-\lambda z)-z}\cdot((\alpha_0+\delta-\alpha(z))p_0-\delta) \qquad (2)$$

$$P^*(z,\theta) = \sum_{n=1}^{\infty} P_n^*(\theta)z^n = \frac{z((\alpha(z)-\alpha_0-\delta)p_0+\delta)}{\lambda+\delta-\theta-\lambda z}\cdot\frac{S^*(\lambda+\delta-\lambda z)-S^*(\theta)}{S^*(\lambda+\delta-\lambda z)-z}. \qquad (3)$$

By inserting $\theta = 0$ in (3), we get

$$P^*(z,0) = \frac{z((\alpha(z)-\alpha_0-\delta)p_0+\delta)}{\lambda+\delta-\lambda z}\cdot\frac{S^*(\lambda+\delta-\lambda z)-1}{S^*(\lambda+\delta-\lambda z)-z} \qquad (4)$$

Let z_r be a root of the equation,

$$(\lambda+\delta-\lambda z_r)\cdot(S^*(\lambda+\delta-\lambda z_r)-z_r) = 0 \qquad (5)$$

then z_r make the denominator of (4) equal to 0. From that, we notice z_r is a root which make the numerator of (4) equal to 0.

$$z_r\cdot((\alpha(z_r)-\alpha_0-\delta)p_0+\delta)\cdot(S^*(\lambda+\delta-\lambda z_r)-1) = 0 \qquad (6)$$

In addition, z_r must be a complex number in $0 < z_r < 1$.

Let z_r be the root which satisfies the equation, $\lambda+\delta-\lambda z_r = 0$, then $z_r = (\lambda+\delta)/\lambda$. Since z_r must be a complex number which satisfies $0 < z_r < 1$, z_r is a root which satisfies the equation, $S^*(\lambda+\delta-\lambda z_r) - z_r = 0$.

We can obtain the shape of the function, $S^*(\lambda+\delta-\lambda z_r)$, from the following result.

$$\frac{\partial}{\partial z_r}S^*(\lambda+\delta-\lambda z_r)=\frac{\partial}{\partial z_r}\int_0^\infty e^{-(\lambda+\delta-\lambda z_r)x}f(x)dx$$

$$=\int_0^\infty \lambda x e^{\lambda z_r x}S(x)dx, \quad (7)$$

which shows us that $\frac{\partial}{\partial z_r}S^*(\lambda+\delta-\lambda z_r)>0$ in the range of $0<z_r<1$. In addition, we can easily obtain that $\frac{\partial^2}{\partial z_r^2}S^*(\lambda+\delta-\lambda z_r)>0$ in $0<z_r<1$. From that, we can determine the shape of the function, $S^*(\lambda+\delta-\lambda z_r)$, and we can see that there is only one root z_r in $0<z_r<1$.

In general $M/G/1$ model, there is a well known equation,

$$B_0^*(\theta)=S^*(\theta+\lambda-\lambda B_0^*(\theta)). \quad (8)$$

Substituting θ in (8) by δ, we can obtain

$$B_0^*(\delta)=S^*(\theta+\lambda-\lambda B_0^*(\delta)). \quad (9)$$

(9) show us that the only root, z_r, is $B_0^*(\delta)$. Inserting $z_r=B_0^*(\delta)$ in the equation (6), we obtain

$$P_0=\frac{\delta}{\alpha_0+\delta-\alpha(B_0^*(\delta))} \quad (10)$$

From (4) and (10), we get

$$P(z)=P_0+P^*(z,0)$$

$$=\frac{\delta}{\alpha_0+\delta-\alpha(B_0^*(\delta))}+$$

$$\frac{z\delta}{\lambda+\delta-\lambda z}\cdot\frac{\alpha(z)-\alpha(B_0^*(\delta))}{\alpha_0+\delta-\alpha(B_0^*(\delta))}\cdot\frac{g^*(\lambda+\delta-\lambda z)-1}{g^*(\lambda+\delta-\lambda z)-z} \quad (11)$$

3. Sojourn Time Distribution

In order to obtain the sojourn time distribution, we use the steady state distribution at an arrival epoch (see Yang et al [3]). If a test customer arrives when there are i customers in the system, his unfinished work is expressed as the summation of the residual of ongoing service time and the i customers' service times. If a test customer arrives during idle periods, he arrives in the group of customers. For that reason, in the case of the test customer who arrives during idle periods, the unfinished work is defined as the summation of the service times of the residual customers of the group, g_R, and his service time. From that, we can describe the unfinished work of an arbitrary test customer as

$$U^*(\theta)=P_g\cdot\frac{1-G(Z)}{(1-z)E(g)}\cdot z\,\Big|_{z=S^*(\theta)}$$

$$+\frac{1-P_g}{1-\pi_0}\cdot\overline{\pi}^*(S^*(\theta),\theta). \quad (12)$$

Since this model has the different customers' arrival rates, PASTA doesn't hold in this model. In other words, the steady state distribution differs from that at an arrival epoch. For that, we should derive the steady state distribution at an arrival epoch.

Let us assume a model having the same steady state distribution at an arrival epoch as the model of this paper, in which PASTA holds. We can easily obtain the model by manipulating the arrival rates, α_k, in idle period as follows: Let $\overline{\alpha}_k$ be the arrival rates in idle periods of the new model. In order that PASTA holds in this model, we have $\overline{\alpha}_0=\lambda$. In addition, we define $\overline{\alpha}_k$ to meet $\overline{\alpha}_k/\overline{\alpha}_0=\alpha_k/\alpha_0$, in order that the steady state distribution at an arrival epoch of the new model is equal to that of the original model. Then

$$\overline{\alpha}_k=\frac{\lambda\alpha_k}{\alpha_0} \quad (13)$$

$$\overline{\alpha}(z)=\sum_{k=1}^{\infty}\overline{\alpha}_k z^k=\frac{\lambda}{\alpha_0}\sum_{k=1}^{\infty}\alpha_k z^k=\frac{\lambda}{\alpha_0}\alpha(z) \quad (14)$$

Using $\overline{\alpha}_0=\lambda$ and (14) instead of α and $\alpha(z)$, we obtain the p_0 and the $P^*(z,\theta)$ of the new model by the formula (3) and (10), which are equal to $\overline{\pi}_0$ and $\overline{\Pi}^*(z,\theta)$ of the new model. From that, we get

$$\overline{\pi}_0=\frac{\delta}{\overline{\alpha}_0+\delta-\overline{\alpha}(B_0^*(\delta))}$$

$$=\frac{\alpha_0\delta}{\lambda\alpha_0+\delta\alpha_0-\lambda\alpha(B_0^*(\delta))} \quad (15)$$

$$\overline{\Pi}^*(z,\theta)$$

$$= \frac{z\left(\left(\bar{\alpha}(z) - \bar{\alpha}_0 - \delta\right)\bar{\pi}_0 + \delta\right)}{\lambda + \delta - \theta - \lambda z} \cdot \frac{S^*(\lambda + \delta - \lambda z) - S^*(\theta)}{S^*(\lambda + \delta - \lambda z) - z}$$

$$= \frac{\lambda z \delta}{\lambda + \delta - \theta - \lambda z} \cdot \frac{\alpha(z) - \alpha\left(B_0^*(\delta)\right)}{\lambda\alpha_0 + \delta\alpha_0 - \lambda\alpha\left(B_0^*(\delta)\right)}$$

$$\cdot \frac{S^*(\lambda + \delta - \lambda z) - S^*(\theta)}{S^*(\lambda + \delta - \lambda z) - z} \qquad (16)$$

Since customers arriving during idle periods enter the system in group size, the $\bar{\pi}_0$ is differ from the probability, P_g. For that reason, we should derive the ratio of customers who enter the system when the system is idle. The ratio is acquired by the number average of the customers. Let $\mathrm{E}(\Gamma)$ be the average number of the customers who arrives during a cycle, a busy period and an idle period. Then

$$P_g = \frac{\mathrm{E}(N_B)}{\mathrm{E}(\Gamma)} = \frac{\mathrm{E}(N_B)}{\mathrm{E}(N_B) + \lambda\mathrm{E}(B)}$$

$$= \frac{\delta\alpha'(1)}{\lambda\alpha_0 + \delta\alpha'(1) - \lambda\alpha\left[B_0^*(\delta)\right]}. \qquad (17)$$

Substituting (17) into (12), we get

$$U^*(\theta) = \frac{\delta S^*(\theta)}{\lambda\alpha_0 + \delta\alpha'(1) - \lambda\alpha\left(B_0^*(\delta)\right)}$$

$$\cdot \frac{(\delta - \theta)\left(\alpha_0 - \alpha\left(S^*(\theta)\right)\right) + \lambda\left(1 - S^*(\theta)\right)\left(\alpha_0 - \alpha\left(B_0^*(\theta)\right)\right)}{\left(\lambda + \delta - \theta - \lambda S^*(\theta)\right)\left(1 - S^*(\theta)\right)} \qquad (18)$$

In $M/G/1$ models with disasters, the sojourn time, t, is described as the $\min(u, D)$. The $T^*(\theta)$ is described as

$$T^*(\theta) = P(u < D) \cdot U^*(\theta \mid u < D) + P(D < u) \cdot D^*(\theta \mid D < u)$$

$$= U^*(\delta) \cdot \frac{U^*(\theta + \delta)}{U^*(\delta)} + \left(1 - U^*(\delta)\right) \cdot \frac{\delta}{\theta + \delta} \cdot \frac{1 - U^*(\theta + \delta)}{1 - U^*(\delta)}. \qquad (19)$$

From (18) and (19), we have

$$T^*(\theta) = \frac{\delta}{\theta + \delta} + \frac{\theta}{\theta + \delta} \cdot \frac{\delta S^*(\theta + \delta)}{\lambda\alpha_0 + \delta\alpha'(1) - \lambda\alpha\left(B_0^*(\delta)\right)}$$

$$\cdot \frac{\lambda\left(1 - S^*(\theta + \delta)\right)\left(\alpha_0 - \alpha\left(B_0^*(\theta)\right)\right) - \theta\left(\alpha_0 - \alpha\left(S^*(\theta + \delta)\right)\right)}{\left(\lambda - \theta - \lambda S^*(\theta + \delta)\right)\left(1 - S^*(\theta + \delta)\right)} \qquad (20)$$

Using (20), we can get the expected sojourn time, W.

$$W = -\frac{d}{d\theta} T^*(\theta)\big|_{\theta=0}$$

$$= \frac{1}{\delta} - \frac{\alpha_0 - \alpha\left(B_0^*(\delta)\right)}{\lambda\alpha_0 + \delta\alpha'(1) - \lambda\alpha\left(B_0^*(\delta)\right)} \cdot \frac{S^*(\delta)}{1 - S^*(\delta)} \qquad (21)$$

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