손실함수에 의한 베이지안 퍼지 가설검정

A Bayesian Fuzzy Hypotheses Testing with Loss Function

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Abstract

We propose some properties of Bayesian fuzzy hypotheses testing by revision for prior possibility distribution and posterior possibility distribution using weighted fuzzy hypotheses $H_0(\theta)$ versus $H_1(\theta)$ on θ with loss function.

Key Words : Bayesian hypotheses, fuzzy hypotheses testing, posterior possibility density, odds ratio, revision of possibility, loss function.

1. Introduction

Bayesian approach to fuzzy Hypotheses testing are frequently used in analysis of subjective concepts. There are some research regarding the Bayesian methods combined with ideas from fuzzy set theory. Schmatter[4] generalized Bayesian methods both for samples of fuzzy data and for fuzzy prior parameters. Taheri et al [5] considered the problem of hypotheses testing when the data(observation) are ordinary(crisp) and the hypotheses are fuzzy, such as : θ is approximately one, θ is very large, and so on. Lapointe et al [2] were studied that this type of inference and to develop the count part of the Bayes rule in the possibilistic framework with the use of conditional possibility distribution.

In this paper we consider the problem of hypotheses testing when the prior distribution and conditional distribution are possibility distribution and the hypotheses are fuzzy by revision probability with loss function.

In Section2, we provide some revision of possibility distribution by Bayesian inference pattern. A Bayesian method for testing fuzzy hypotheses with loss function are given by Section 3. Some example are presented in Section 4.

2. Rivision of possibility distribution

The possibility distribution of variable X taking values in U is a function from U to

[0,1] and is denoted $\pi_x(U)$. The joint possibility distribution of the variable X (taking values in U) and Y (taking values in V) is denoted $\pi_{(X,Y)}(U,V)$ and represented the possibility that $X=v$ and $Y=u$. The conditioned possibility distribution if U given $X=v$ is devoted $\pi_{Y|X}(V|U)$.

2.1 Triangular norms and conorms

A triangular norm is a function T from $[0,1]\times[0,1]$ to $[0,1]$ that satisfies the following conditions :

$\forall a,\ b,\ a',\ b',c\in[0,1]$

(i) $T(a,b)=T(b,a)$,

(ii) If $a\leq a'$ and $b\leq b'$ then $T(a,b)\leq T(a'b')$,

(iii) $T(T(a,b),c)=T(a,T(b,c))$,

(iv) $T(0,1)=a$.

A triangular conorm is a function S from $[0,1]\times[0,1]$ to $[0,1]$ that satisfies conditions (i)-(iii) above in addition to the following are :

(v) $S(a,0)=a$.

We also use the following notation :

$S_u[f(u)]=S(\{f(u): u\in U\})$ where f is a function from U to $[0,1]$. For any t-norm T , there is a dual t-conorm S defined by

$S(a,b)=1-T(1-a,1-b)$. (2-1)

We cite some interesting t-norms together with dual t-conoms :

$T(a,b)=Min(a,b)$ $S(a,b)=Max(a,b)$ or $Sup(a,b)$. (2-2)

Given $\pi_{Y|X}(v|u)$ and $\pi_X(v)$, $\pi_Y(u)$ is defined by

$\pi_Y(v)=S_U[T(\pi_X(u),\ \pi_{Y|X}(v|u)]$. (2-3)

For the shake of clarity, the following notation will be used to represent Eg.(2-3)

$\pi_Y(v)=\pi_X(u)\cdot\ \pi_{Y|X}(v|u)$. (2-4)

The joint probability distribution of X and Y is simple obtained by combining one of the conditional possibility distribution with the marginal possibility distribution of the appropriate variable. There are two different way to derive it :

$\pi_{(X,Y)}(u,v)=T(\pi_X(u),\ \pi_{Y|X}(v|u))$,(2-5)

satisfies $\pi_{(X,Y)}(u,v)=Min(\mu_A(u),\mu_B(v))$.

2.2 Revision of possibility distribution

T is any t-norm and S is any t-conorm, when there is no confusion the possibility distribution. $\pi_X(v)$, $\pi_Y(u)$, $\pi_{Y|X}(v|u)$ and $\pi_{(X,Y)}(u,v)$ will be represented by, $\pi(u)$, $\pi(v)$, $\pi(v|u)$ and $\pi(u,v)$ respectively.

Theorem 2.1. If $\pi(u)$ and $\pi(v|u)$ are both normal, then $\pi(u,v)$ and $\pi(v)$ defined by

$\pi(u,v)=T(\pi(v|u),\pi(u))$, (2-6)

$\pi(v)=\ S_U[T(\pi(v|u),\pi(u))]=S_U[\pi(u,v)]$

are normal.

Theorem 2.2. The equality

$\pi(u)=\ S_V[T(\pi(v|u),\pi(u))]=\ S_V[\pi(u,v)]$, $\forall u\in U$ (2-7)

holds for any $\pi(v|u)$ normal, for any $\pi(u)$, and for any t-conorm T iff $S=Sup$.

Now, let us define the operation λ_T as

$\lambda_T(a,c)=Sup\{x\in[0,1]: T(a,x)=c\}$. (2-8)

Theorem 2.3. The least specific solution to Bayesian methods always exists and is defined by

$\pi_1(u|v)=\ \lambda_T(\pi(v),\pi(u,v))$. (2-9)

Since a t-norm is non decreasing(monotonicity property), we can also write

$$\pi_1(u|v) = \psi_T(\pi(v), \pi(u, v)). \qquad (2\text{-}10)$$

Using Eg(2-10) and operator ψ_T, we derive some rules of inference for specific t-norm T. To simplify rules, let

$$T(u, v) = T(\pi(u), \pi(v|u))$$

and $\pi(v) = \pi(u) \cdot \pi(v|u)$ where $\pi(u) \geq \pi(u, v)$, we obtain the following rule for the t-norm :

Rule : $T(a, b) = Min(a, b)$ (2-12)

$$\pi(u|v) = \begin{cases} 1 & \text{if } \pi(v) = \pi(u, v) \\ \pi(u, v) & \text{if } \pi(v) > \pi(u, v). \end{cases} \qquad (2\text{-}13)$$

The fuzzy null hypothesis and the fuzzy alternative hypothesis can be defined as follows.

3. Bayesian approach to loss function

Let $X = (X_1, \cdots, X_n)$ be random sample, where X_i has the p.d.f. $\pi(x_i|\theta)$ with unknown $\theta \in \Theta$, whose prior density is $\pi(\theta)$ where Θ is parameter space. Suppose that two membership functions $H_o(\theta)$ and $H_1(\theta)$ are given.

We want to test:

$$H_0 : \theta \text{ is } H_o(\theta),$$

$$H_1 : \theta \text{ is } H_1(\theta).$$

on the basic of a Bayesian method.

Consider a prior possibility density $\pi(\theta)$ for θ and assume that $\pi(x|\theta)$ is the conditional possibility density of X with $\theta \in \Theta$. The conditional possibility density θ for given $X = x$, is called the posterior possibility density of θ and dented by $\pi(\theta|x)$. The following two relative are well known :

$$\pi(\theta|x) \propto \pi(\theta) \cdot \pi(x|\theta), \qquad (3\text{-}1)$$

$$\pi(x, \theta) = \pi(\theta) \cdot \pi(x|\theta) = \pi_X(x) \cdot \pi(\theta|x). \quad (3\text{-}2)$$

Consider the problem of testing the fuzzy hypotheses $H_0(\theta)$ versus $H_1(\theta)$, based a random sample from possibility density $\pi(x|\theta)$ with prior possibility density $\pi(\theta)$ for θ .

Definition 3.1. A Bayes test reject H_0 iff the posterior possibility density under H_0. which is weighted by $H_0(\theta)$ on θ is less than the posterior possibility density under $H_1(\theta)$, which is weighted by $H_1(\theta)$ on θ, i.e. if

$$\int \pi(\theta|x)\, H_0(\theta) d\theta < \int \pi(\theta|x)\, H_1(\theta) d\theta. \quad (3\text{-}3)$$

We consider a loss function on the basis of the membership functions $H_0(\theta)$ and $H_1(\theta)$.

Definition 3.2. For testing the hypotheses " $H_0 : \theta$ is $H_0(\theta)$" versus " $H_1 : \theta$ is $H_1(\theta)$", we define the following loss function:

$$L(\theta, H_0) = a(\theta)[1 - H_0(\theta)], \qquad (3\text{-}4)$$

$$L(\theta, H_1) = b(\theta)[1 - H_1(\theta)],$$

where $a(\theta)$ and $b(\theta)$ are two arbitrary nonnegative function.

Choosing $a(\theta)$ and $b(\theta)$ depends on our sensitivity to false rejection or false acceptance.

Theorem 3.1. If

$$L(\theta, H_0) = a(\theta)[1 - H_0(\theta)],$$

$$L(\theta, H_1) = b(\theta)[1 - H_1(\theta)],$$

then the Bayses test accepts H_0 iff

$$\int a(\theta)[1 - H_0(\theta)]\pi(\theta|x) d\theta$$

$$\leq \int b(\theta)[1 - H_1(\theta)]\pi(\theta|x) d\theta. \qquad (3\text{-}5)$$

4. An example

We present a example to illustrate the application. Let we have prior possibility distribution $\pi(\theta)$:

$$\pi(\theta)=\begin{cases}\frac{\theta-2}{3}, & \text{if } 2<\theta<5,\\ 1, & \text{if } 5\le\theta\le 8,\\ \frac{10-\theta}{2}, & \text{if } 8<\theta<10,\\ 0 & \text{otherwise.}\end{cases} \quad (4\text{-}1)$$

and conditional possibility distribution $\pi(x|\theta)$:

$$\pi(x|\theta)=\begin{cases}\frac{3x-6}{\theta}-1 & \frac{\theta}{3}+2<x\le\frac{2\theta}{3}+2,\\ \frac{6-3x}{\theta}+3 & \frac{2\theta}{3}+2<x<\theta+2,\\ 0 & \text{otherwise.}\end{cases} \quad (4\text{-}2)$$

Thus, we illustrate the application of the possibility process of forecasts for different rules of inference, with Rule $T=Min(a,b)$, We obtain using Eq(2-13)

$$\pi(\theta|4)=\begin{cases}\frac{\theta-2}{3} & \text{if } 2<\theta<3.772,\\ 1 & \text{if } \theta=3.772,\\ \frac{6}{\theta}-1 & \text{if } 3.772<\theta<6,\\ 0 & \text{otherwise.}\end{cases} \quad (4\text{-}3)$$

Now we want to test

H_0 : $\theta\simeq 3$

H_1 : $\theta\simeq 4$

where the membership function

$$H_0(\theta)=\begin{cases}\theta-2 & \text{if } 2<\theta\le 3,\\ -\theta+4 & \text{if } 3<\theta\le 4,\\ 0 & \text{otherwise,}\end{cases}$$

$$H_1(\theta)=\begin{cases}\theta-3 & \text{if } 3<\theta\le 4,\\ -\theta+5 & \text{if } 4<\theta\le 5,\\ 0 & \text{otherwise.}\end{cases}$$

From Eq(3-5), if we have $a(\theta)=b(\theta)$ then

$$1-\int H_0(\theta)\pi(\theta|x)d\theta$$

$$=1-\int_2^3\frac{\theta-2}{3}(\theta-2)d\theta + \int_3^{3.772}\frac{\theta-2}{3}(-\theta+4)d\theta + \int_{3.772}^4(\frac{6}{\theta}-1)(-\theta+4)d\theta=0.4627$$

$$=0.5373.$$

$$1-\int H_1(\theta)\pi(\theta|x)d\theta$$

$$=1-\int_3^{3.772}\frac{\theta-2}{3}(\theta-3)d\theta + \int_{3.772}^4(\frac{6}{\theta}-1)(\theta-3)dx + \int_4^5(\frac{6}{\theta}-1)(-\theta+5)d\theta=0.4544$$

$$=0.5456$$

Thus we accept the $H_0(\theta)$ with the odds ratio

$$1-\int H_0(\theta)\pi(\theta|x)d\theta\le 1-\int H_1(\theta)\pi(\theta|x)d\theta.$$

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