E_N^n 상의 비선형 퍼지 미분방정식에 대한 대역해의 존재성

The existence of a global solution for the nonlinear fuzzy differential equations in E_N^n.

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Abstract

This paper we study the existence of a global solution for the nonlinear fuzzy differential equations in E_N^n by using the concept of fuzzy number of dimension n whose values are normal, convex, upper semicontinuous and compactly supported surface in R^n.

Keywords and Phrases : fuzzy number of dimension n, nonlinear fuzzy differential equations

1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported fuzzy sets in R^n. Seikkala [5] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases}$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number in E^1. Diamond and Kloeden [2] proved the fuzzy optimal control for the following system:

$$\begin{cases} \dot{x}(t) = a(t)x(t) + u(t), \\ x(0) = x_0 \end{cases}$$

where $x(\cdot)$, $u(\cdot)$ are nonempty compact interval-valued functions on E^1.

We consider the existence and uniqueness of solution for
the following nonlinear fuzzy differential equations:

$$\begin{cases} \dot{x}(t) = a(t)x(t) + f(t, x(t)), \\ x(0) = x_0, \end{cases}$$

where $a: [0, T] \to E_N^n$ is fuzzy coefficient, initial value $x_0 \in E_N^n$ and $f: [0, T] \times E_N^n \to E_N^n$ is nonlinear function.

Let E_N^n be the set of all fuzzy numbers in R^n with edges having bases parallel to axis $X_1, \cdots, X_n$.

For example, E_N^2 be the set of all fuzzy pyramidal numbers in R^2 with edges having rectangular bases parallel to the axis X_1 and X_2 [4].

2. Properties of fuzzy numbers

In this section, we give some definitions, properties and notations of the fuzzy number of dimension n.

Definition 2.1. We consider a fuzzy graph $G \subset R^n$ that is a functional fuzzy relation in R^n such that its membership function $\mu_G(x_1, \cdots, x_n) \in [0,1]$, $(x_1, \cdots, x_2) \in R^n$ has the following properties:

1. For all $x_i \in R$, $(i=1, \cdots, n)$,

$$\mu_G(x_1, \cdots, x_i, \cdots, x_n) \in [0,1]$$

is a convex membership function.

2. For all $\alpha \in [0,1]$,

$$\{(x_1, \cdots, x_n) \in R^n : \mu_G(x_1, \cdots, x_n) = \alpha\}$$

is a convex set.

3. There exists $(x_1, \cdots, x_n) \in R^n$,

$$\mu_G(x_1, \cdots, x_n) = 1 .$$

If the above conditions are satisfied, the fuzzy subset G is called a fuzzy number of dimension n.

The first projection of G is

$$\vee_{\{x_2, \cdots, x_n\}} \mu_G(x_1, \cdots, x_n) = \mu_{A_1}(x_1),$$

the second projection of G is

$$\vee_{\{x_1, x_3, \cdots, x_n\}} \mu_G(x_1, \cdots, x_n) = \mu_{A_2}(x_2)$$

and the i-th projection of G is

$$\vee_{\{x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_n\}} \mu_G(x_1, \cdots, x_n) = \mu_{A_i}(x_i),$$

$(i=3, \cdots, n)$.

We denote by fuzzy number in

$$E_N^n \ A = (a_1, a_2, \cdots, a_n),$$

where a_i is projection of A to axis X_i $(i=1, \cdots, n)$, respectively.

And a_i $(i=1, \cdots, n)$s fuzzy number in R.

Definition 2.2. The α-level set of fuzzy number in E_N^n is defined by

$$[A]^\alpha = \{(x_1, \cdots, x_n) \in R^n : (x_1, \cdots, x_n) \in \prod_{i=1}^{n} [a_i]^\alpha\},$$

where notation $\prod$ is the Cartesian product of sets.

Definition 2.3. Let A and B in E_N^n, for all $\alpha \in (0,1]$,

(2.1) $$A = B \Leftrightarrow [A]^\alpha = [B]^\alpha .$$

(2.2) $$[A *_n B]^\alpha = \prod_{i=1}^{n} [a_i * b_i]^\alpha ,$$

where $*_n$ is operation in E_N^n and $*$ is operation in E_N.

Definition 2.4. The derivative $x'(t)$ of a fuzzy process $x \in E_N^n$ is defined by

$$[x'(t)]^\alpha = \prod_{i=1}^{n} [(x_{il}^\alpha)'(t), (x_{ir}^\alpha)'(t)], \ 0 < \alpha \leq 1$$

provided that is equation defines a fuzzy $x'(t) \in E_N^n$.

The fuzzy integral $\int_a^b x(t)dt$, $a, b \in I$ is defined by

$$[\int_a^b x(t)dt]^\alpha = \prod_{i=1}^{n} [\int_a^b x_{il}^\alpha(t)\,dt, \int_a^b x_{ir}^\alpha(t)\,dt]$$

provided that the Lebesgue integrals on the right exist.

Let $\prod_{i=1}^{n} [a_i]^\alpha$, $0 < \alpha \leq 1$, be a given family of nonempty areas.

If

(2.3) $\prod_{i=1}^{n} [a_i]^\beta \subset \prod_{i=1}^{n} [a_i]^\alpha$ for $0 < \alpha < \beta < 1$ and

(2.4) $\prod_{i=1}^{n} \lim_{k \to \infty} [a_i]^{\alpha_k} = \prod_{i=1}^{n} [a_i]^\alpha$

whenever (α_k) is a nondecreasing sequence converging to $\alpha \in (0,1]$, then the family $\prod_{i=1}^{n} [a_i]^\alpha$, $0 < \alpha \leq 1$, represents the α-level sets of a fuzzy number $A \in E_N^n$.

Conversely, if $\prod_{i=1}^{n} [a_i]^\alpha$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number in R^n, then the conditions (2.3) and (2.4) hold true.

We define the metric d_∞ on E_N^n.

Definition 2.5. Let A, $B \in E_N^n$.

$$\begin{aligned} d_\infty(A,B) &= \sup\{ d_H([A]^\alpha, [B]^\alpha) : \alpha \in (0,1]\} \\ &= \sup\{ d_H(\prod_{i=1}^{n} [a_i]^\alpha, \prod_{i=1}^{n} [b_i]^\alpha) : \alpha \in (0,1]\} \\ &= \sup\{ \sqrt{\sum_{i=1}^{n} (d_H([a_i^\alpha], [b_i^\alpha]))^2} : \alpha \in (0,1]\} \end{aligned}$$

where d_H is the Hausdorff distance.

The supremum metric H on $C([0, T]: E_N^n)$ is defined by

$$H(x, y) = \sup\{d_\infty(x(t), y(t)): t \in [0, T]\}$$

for all $x, y \in C([0, T]: E_N^n)$.

3. Existence of a global solution

In this section, we show the classical existence theorem for the following initial value problems assumes that the mapping f satisfies a global Lipschitz condition:

$$(3.1) \qquad \begin{cases} \dot{x}(t) = a(t)x(t) + f(t, x(t)), \\ x(0) = x_0, \end{cases}$$

with fuzzy coefficient $a: [0, T] \to E_N^n$, initial value $x_0 \in E_N^n$ and inhomogeneous term $f: [0, T] \times E_N^n \to E_N^n$ satisfies a global Lipschitz condition.

Assume that the following hypotheses:

(H1) Inhomogeneous term $f: [0, T] \times E_N^n \to E_N^n$ satisfies a global Lipschitz condition, there exists a finite constant $k_i > 0$ such that

$$(3.2) \qquad \begin{aligned} & d_H([f_i(s, x(s))]^\alpha, [f_i(s, y(s))]^\alpha) \\ & \leq k_i \, d_H([x_i(s)]^\alpha, [y_i(s)]^\alpha) \end{aligned}$$

for all $x_i(s)$, $y_i(s) \in E_N$ and $f_i: [0, T] \times E_N \to E_N$ $(i = 1, \cdots, n)$ is the i-th projection of f.

We denote $k = \max\{k_i | i = 1, \cdots, n\}$.

Theorem 3.1. For any finite $T > 0$ and $x_0 \in E_N^n$, a fuzzy differential equation (3.1) on (E_N^n, d_∞) which satisfies a global Lipschitz condition (3.2) has a unique solution $x: [0, T] \to E_N^n$ corresponding to the initial value $x(0) = x_0$.

Proof. Choose $\eta > 0$ so $\frac{T}{\eta}$ is an integer, L say, and $ck\eta < 1$ where k is the constant in the global Lipschitz condition. For each $\xi \in E_N^n$ and $t \in [0, \eta]$ define $(G_0 \xi)(t) \in E_N^n$ by

$$(G_0 \xi)(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds$$

where $S(t)$ is fuzzy number of dimension n and

$$[S(t)]^\alpha = \prod_{i=1}^n [S_i(t)]^\alpha = \prod_{i=1}^n [S_{il}^\alpha(t), \; S_{ir}^\alpha(t)]$$

where $S_{il}^\alpha(t)$ is $\exp\{\int_0^t a_l^\alpha(s)ds\}$ and $S_{ir}^\alpha(t)$ is $\exp\{\int_0^t a_r^\alpha(s)ds\}$.

And $S_{ij}^\alpha(t)$ $(j = l, r)$ is continuous.

That is, there exists a constant $c > 0$ such that $|S_{ij}^\alpha(t)| \leq c$ for all $t \in [0, T]$.

Then $G_0\xi: [0, \eta] \to E_N^n$ is continuous, so G_0 is a mapping from $C([0, \eta]: E_N^n)$ into itself.

By the global Lipschitz condition,

$$\begin{aligned} & d_\infty((G_0x)(t), (G_0y)(t)) \\ & = \sup_{\alpha \in (0,1]} d_H([\int_0^t S(t-s)f(s, x(s))ds]^\alpha, \\ & \qquad [\int_0^t S(t-s)f(s, y(s))ds]^\alpha)ds \\ & \leq ck \int_0^t \sup_{\alpha \in (0,1]} d_H([x(s)]^\alpha, [y(s)]^\alpha)ds. \end{aligned}$$

Hence

$$\begin{aligned} & H(G_0x, G_0y) \\ & = \sup_{t \in [0, \eta]} d_\infty((G_0x)(t), (G_0y)(t)) \\ & \leq ck\eta \sup_{t \in [0, \eta]} d_\infty([x(t)]^\alpha, [y(t)]^\alpha) \\ & = ck\eta H(x, y) \end{aligned}$$

for all $x, y \in C([0, \eta]: E_N^n)$. Since $ck\eta < 1$ the mapping G_0 is a contraction and thus, by the Banach Contraction Mapping theorem, has a unique fixed point $x^{(0)} = G_0 x^{(0)} \in C([0, \eta]: E_N^n)$, that is with

$$x^{(0)}(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds$$

for $t \in [0, \eta]$. Obviously, $x^{(0)}(0) = x_0$ here. Now define $I_l = [l\eta, (l+1)\eta]$ for

$l=1,2,\cdots,L-1$.

We repeat the above argument for the mapping G_l from $C(I_l : E_N^n)$ into itself defined by

(3.3)

$$(G_l\xi)(t)=x^{(l-1)}(l\eta)+\int_{l\eta}^{t}S(t-s)f(s,\xi(s))ds$$

for all $\xi\in C(L_l : E_N^n)$ and $t\in I_l$ to obtain a unique fixed point $x^{(l)}=G_l x^{(l)}\in C(I_l : E_N^n)$, that is satisfying

(3.4)

$$x^{(l)}(t)=x^{(l-1)}(l\eta)+\int_{l\eta}^{t}S(t-s)f(s,x^{(l)}(s))ds$$

for $t\in I_l$. Finally we define $x:[0,T]\rightarrow E_N^n$ by $x(t)\equiv x^{(l)}(t)$ for $t\in I_l$.

By its construction, x is continuous and satisfies

$$(3.5)\quad x(t)=S(t)x_0+\int_0^t S(t-s)f(s,x(s))ds$$

for all $t\in[0,T]$, and is the only such function satisfying (3.5). Thus x is the unique solution of the given initial value problem.

4. Example

In this section, as an example, we cite the existence of a global solution for the nonlinear fuzzy differential equations in E_N^2.

Example 4.1. Consider the fuzzy solution of the nonlinear fuzzy differential equations in E_N^2 :

$$(\text{F.D.E})\begin{cases}\dot{x}=\tilde{2}\ x+\tilde{2}\ t\ x^2, & 0\le t\le T,\\ x(0)=\tilde{2}\ \in E_N^2.\end{cases}$$

Let $f_i(t,x(t))=\tilde{2}\ t\ x_i(t)^2$ be the i-th projection of $f(t,x(t))$, where $x_i(t)$ is the i-th projection of $x(t)$.

Let $k_i=3T\ |x_{ir}^{\alpha}(t)+y_{ir}^{\alpha}(t)|>0,\ (i=1,2)$.

Then α-level set of $f_i(t,x(t))$ is

$$\begin{aligned}&[f_i(t,x(t))]^{\alpha}\\ &=[\tilde{2}\ t\ x_i(t)^2]^{\alpha}=t[\tilde{2}]^{\alpha}\cdot[x_i(t)^2]^{\alpha}\\ &=t[\alpha+1,3-\alpha]\cdot[(x_{il}^{\alpha}(t))^2,(x_{ir}^{\alpha}(t))^2]\\ &=t[(\alpha+1)(x_{il}^{\alpha}(t))^2,(3-\alpha)(x_{ir}^{\alpha}(t))^2]\end{aligned}$$

where $[x_i(t)]^{\alpha}=[x_{il}^{\alpha}(t),x_{ir}^{\alpha}(t)]$ and $[\tilde{2}]^{\alpha}=[\alpha+1,3-\alpha]$ for all $\alpha\in[0,1]$.

Therefore

$$\begin{aligned}&d_H([f_i(t,x(t))]^{\alpha},[f_i(t,y(t))]^{\alpha})\\ &=d_H(t[(\alpha+1)(x_{il}^{\alpha}(t))^2,(3-\alpha)(x_{ir}^{\alpha}(t))^2],\\ &\qquad t[(\alpha+1)(y_{il}^{\alpha}(t))^2,(3-\alpha)(y_{ir}^{\alpha}(t))^2])\\ &=t\max\{(\alpha+1)\,|(x_{il}^{\alpha}(t))^2-(y_{il}^{\alpha}(t))^2|,\\ &\qquad (3-\alpha)\,|(x_{ir}^{\alpha}(t))^2-(y_{ir}^{\alpha}(t))^2|\}\\ &\le T(3-\alpha)\max\{|x_{il}^{\alpha}(t)-y_{il}^{\alpha}(t)|\ |x_{il}^{\alpha}(t)+y_{il}^{\alpha}(t)|,\\ &\qquad |x_{ir}^{\alpha}(t)-y_{ir}^{\alpha}(t)|\ |x_{ir}^{\alpha}(t)+y_{ir}^{\alpha}(t)|\}\\ &\le 3T\,|x_{ir}^{\alpha}(t)+y_{ir}^{\alpha}(t)|\\ &\qquad \max\{|x_{il}^{\alpha}(t)-y_{il}^{\alpha}(t)|,|x_{ir}^{\alpha}(t)-y_{ir}^{\alpha}(t)|\}\\ &=k_i\,d_H([x_i(t)]^{\alpha},[y_i(t)]^{\alpha}).\end{aligned}$$

Since f satisfies a global Lipschitz condition, from Theorem 3.1 (F.D.E) has a unique fuzzy solution.

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