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퍼지 수 공간의 컴팩트 볼록 집합에 관한 연구

On compact convex subsets of fuzzy number space

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ABSTRACT

By Mazur's theorem, the convex hull of a relatively compact subset a Banach space is also relatively compact. But this is not true any more in the space of fuzzy numbers endowed with the Hausdorff-SKorohod metric.

In this paper, we establish a necessary and sufficient condition for which the convex hull of K is also relatively compact when K is a relatively compact subset of the space $F(R^k)$ of fuzzy numbers of R^k endowed with the Hausdorff-SKorohod metric.

Keywords : Fuzzy numbers, Convex hulls, Relatively compact subsets, the Hausdorff-SKorohod metric

1. Introduction

Let $F(R^k)$ be the space of fuzzy numbers in the Euclidean space R^k. i.e., the family of all normal, fuzzy convex, upper-semicontinuous and compactly supported fuzzy sets in R^k. Even though the addition and scalar multiplication on $F(R^k)$ are defined as usual, $F(R^k)$ is not a vector space since the additive does not exist. Nevertheless, we can define the concept of convexity on $F(R^k)$ as in the case of a vector space. That is, $A \subset F(R^k)$ is said to be convex if $\lambda u+(1-\lambda)v \in A$ whenever $u, v \in A$ and $0 \le \lambda \le 1$.

Also, the closed convex hull $co(A)$ of $A \subset F(R^k)$ is defined to be the intersection of all convex subsets of $F(R^k)$ that contains A. Then as in the case of a vector space, we can easily show that $co(A)$ is equal to the family of consisting of all fuzzy numbers in the form $\lambda_1 u_1 + \cdots + \lambda_n u_n$, where $u_1, \ldots, u_n$ are any elements of A, $\lambda_1, \ldots, \lambda_n$ are non-negative real numbers satisfying $\sum_{i=1}^{n} \lambda_i = 1$ and $n = 2.3, \ldots$.

Using this concept, Kim [7] obtained a criteria for which $co(K)$ is also relatively compact when K is a relatively compact subset of $F(R^k)$ endowed the Hausdorff-Skorohod metric. The purpose of this paper is to establish another criteria as a continuation of Kim [7].

2. Preliminaries

Let $P(R^k)$ be the family of all non-empty compact and convex subsets of R^k. Then $P(R^k)$ is metrizable by the Hausdorff metric h defined by

$$h(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\},$$

where $|\cdot|$ is the usual norm in R^n.

It is well known that the metric $(P(R^n),h)$ is complete and separable (See Debreu [4]). The addition and scalar multiplication in $P(R^k)$ are defined as usual;

$$A+B=\{a+b \mid a\in A, b\in B\},$$
$$\lambda A=\{\lambda a \mid a\in A\}.$$

Let $F(R^k)$ denote the space of fuzzy numbers, i.e., the family of all normal, fuzzy convex and upper-semicontinuous fuzzy sets u in R^k such that

$$supp\ u = cl\ \{x\in R^k : u(x) > 0\}$$

is compact. For a fuzzy set u in R^k, we define the α- level set of u by

$$[u]^\alpha = \begin{cases} \{x : u(x) \geq \alpha\}, & 0<\alpha\leq 1, \\ supp\ u, & \alpha=0. \end{cases}$$

Then it follows that $u \in F(R^k)$ if and only if $[u]^\alpha \in P(R^k)$ for each $\alpha \in [0,1]$.

Lemma 2.1. For $u \in F(R^n)$, let us define $f_u : [0,1] \to ((P(R^k),h)$ by $f_u(\alpha) = [u]^\alpha$.

Then (1) f_u is non-increasing; i.e., $\alpha \leq \beta$ implies $f_u(\alpha) \supset f_u(\beta)$.

(2) f_u is left-continuous on $(0,1]$.

(3) f_u has right-limits on $[0,1)$ and is right-continuous at 0.

Conversely, if $g : [0,1] \to ((P(R^k),h)$ is a function satisfying the above conditions (1)-(3), then there exists a unique $v \in F(R^k)$ such that

$$g(\alpha) = [v]^\alpha \text{ for all } \alpha \in [0,1].$$

Proof: See Kim [7].

If we denote the right-limit of f_u at $\alpha \in [0,1)$ by $L_{\alpha^\cdot} u$, then

$$L_{\alpha^\cdot} u = cl\{x\in R^k : u(x) > \alpha\}.$$

Thus, if we define $j_u(\alpha) = h(L_\alpha u, L_{\alpha^\cdot} u)$, then the function f_u is continuous at α if and only if $j_u(\alpha) = 0$.

The addition and scalar multiplication in $F(R^k)$ are defined as usual;

$$(u+v)(x) = \sup_{y+z=x}\min(u(y), v(z)),$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda), & \text{if } \lambda\neq 0 \\ I_{\{0\}}(x), & \text{if } \lambda=0 \end{cases}$$

where $I_{\{0\}}$ is the indicator function of $\{0\}$.

Lemma 2.2. For each $u \in F(R^k)$ and $\varepsilon > 0$, there exists a partition $0=\alpha_1<\alpha_2<\ldots<\alpha_r=1$ of $[0,1]$ such that $h(L_{\alpha_{i-1}} u, L_{\alpha_i} u) < \varepsilon$ for all $i=1,2,\ldots,r$.

Proof. See Joo and Kim [6].

The above lemma implies that $J_u(\varepsilon) = \{\alpha \mid j_u(\alpha) > \varepsilon\}$ is finite for each $u\in F(R^k)$ and $\varepsilon > 0$. Now, we define the metric d_∞ on $F(R^k)$ by

$$d_\infty(u,v) = \sup_{0\leq\alpha\leq 1} h([u]^\alpha, [v]^\alpha).$$

Also, the norm of $u\in F(R^k)$ is defined as

$$\| u \| = d_\infty(u, I_{\{0\}}) = \sup_{x\in L_0 u} |x|.$$

Then it is well-known that $(F(R^k), d_\infty)$ is complete, but is not separable. (See Klement et al. [8])

Recently, Joo and Kim [5,6] introduced a new metric on $F(R^k)$ which makes it a separable metric space as follows:

Definition 2.3. Let T be the class of strictly increasing continuous mappings of $[0,1]$ onto itself. For $u, v \in F(R^k)$, we define

$$d_s(u, v) = \inf \{\varepsilon > 0: \text{there exists a } t \in T \text{ such that } \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \varepsilon \text{ and } d_\infty(u, t(v)) \le \varepsilon \},$$

where $t(v)$ denotes the composition of v and t.

It follows immediately that d_s is a metric on $F(R^k)$ and $d_s(u, v) \le d_\infty(u, v)$. The metric d_s will be called the Hausdorff-Skorohod metric.

3. Main Results

Through this section, we assume that the space $F(R^k)$ is endowed the Hausdorff-Skorohod metric topology. Let us denote by $C(F(R^k))$ the collection of all relatively compact subsets K for which $co(K)$ is also relatively compact. then the result established by Kim [7] is as follows;

Theorem 3.1. Let K be a relatively compact subset of $F(R^k)$. Then

$K \in C(F(R^k))$ if and only if

$$S_\varepsilon(K) = \{\alpha \in (0,1) \mid \sup_{u \in K} j_u(\alpha) > \varepsilon\}$$

is finite for every $\varepsilon > 0$.

We start with some results which can be obtained as an application of theorem 3.1. For $A \subset F(R^k)$ and $\varepsilon > 0$, let

$$A_\varepsilon = \{u \in A \mid \sup_{\alpha \in (0,1)} j_u(\alpha) > \varepsilon\}.$$

Lemma 3.2. If A_ε is finite, then $S_\varepsilon(A)$ is finite.

Theorem 3.3. If K is a relatively compact subset of $F(R^k)$ and K_ε is finite for every $\varepsilon > 0$, then

$$K \in C(F(R^k)).$$

Theorem 3.4. Let K be a relatively compact and convex subset of $F(R^k)$. If $\{u_n\}$ is a sequence of K, then for some $u_0 \in F(R^k)$,

$$\lim_{n \to \infty} d_s(u_n . u_0) = 0 \text{ if and only if } \lim_{n \to \infty} d_\infty(u_n . u_0) = 0.$$

Corollary 3.5. If $K \in C(F(R^k))$, then K is relatively compact in the d_∞-metric topology.

Now we show that a characterization of relatively compact sets in $F(R^k)$ obtained by Joo and Kim [6] can be sharpened considerably if we restrict our attention to $C(F(R^k))$. The following two lemmas are needed in the proof.

Lemma 3.6. Let $\{u_n\}$ be a sequence in $F(R^k)$ such that $\lim_{n \to \infty} d_s(u_n . u_0) = 0$ for some $u_0 \in F(R^k)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \alpha_0$$

and $h(L_{\alpha_n} u_n, L_{\beta_n} u_n) > \varepsilon > 0$ for sufficiently large n, then $j_{u_0}(\alpha_0) \ge \varepsilon$.

Lemma 3.7. Let $K \in C(F(R^k))$ and $\{u_n\}$ be a sequence in $F(R^k)$ such that

$$\lim_{n \to \infty} d_s(u_n . u_0) = 0 \text{ for some } u_0 \in F(R^k).$$

Then $j_{u_0}(\alpha) > \varepsilon$ implies $\alpha \in S_\varepsilon(K)$.

Theorem 3.8. Let K be a subset of $F(R^k)$. Then $K \in C(F(R^k))$ if and only if the following two conditions hold;

(1) $\sup\{\|u\| : u \in K\} < \infty$.

(2) For each $\varepsilon > 0$, there exists a partition $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_r = 1$ of $[0,1]$ such that

$$\sup_{u \in K} h(L_{\alpha_{i-1}} u, L_{\alpha_i} u) < \varepsilon$$

for all $i = 1, 2, \ldots, r$.

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