Enhanced Fractal Coding Method Using Candidate Transformations

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Abstract

This paper presents a new fractal coding scheme to find more optimal transformation by estimation of the optimal attractor. The conventional fractal coding schemes based on the collage theorem obtain the transformation to minimize the distance between an original image and its collage image. Heavy computation is why the schemes widely adopt the theorem. In other words, the optimal transformation can be obtained after the attractors of all the possible transformations are generated and then compared with an original image. It is clear that this process is not practical. Therefore, we introduce a sub-optimal scheme that provides better transformation than the conventional scheme, relieving the complexity problem in the optimal transformation. In a simple case, the optimal transformation can be obtained considering all the attractors and then our scheme is compared with the optimal. In general cases not to be able to find the optimal, our scheme is also evaluated and compared with the conventional schemes.

I. Introduction

The main idea of fractal image coding is to represent an image as a transformation to generate it. This is based on the mathematics of the iterative function system (IFS) and the collage theorem introduced by Barnsley [1]. Since Jacquin [2] has first introduced the automated fractal image coding scheme, many researchers have improved its performance.

As the performance of fractal coders is determined by the distance between an original image and an attractor of a transformation, it is important to obtain the transformation to generate the attractor as close to the original image as possible. We can define the optimal transformation as a transformation to minimize the distance. However we have a difficulty in finding the optimal transformation due to heavy computation and it is practically impossible to implement. Thus the conventional fractal coding schemes limit just the upper bound of the distance between an original image and an attractor, minimizing the distance between an original image and its collage image instead of the attractor. As these schemes are not direct approach to an attractor, they have a disadvantage that the bound is not tight. Honda [3] proposed such a method that the bound is tighter using the extended collage theorem and Harada [4] had an effort to obtain more optimal transformation using the domain pool updated by already coded blocks. Since the implementation of Honda's method is nearly identical with that of optimal coding method, the method is available for very simple cases, but not for general cases.

In this paper, we propose a coding scheme, which is applicable to general cases, to find more optimal transformation. Our scheme is based on the fact that good estimation of the optimal attractor gives more optimal transformation.

II. Background and Optimal Transformation

Let (X,d) denote a metric space of digital images, where $d(\cdot,\cdot)$ is the metric. Consider a transformation w:$X \to X$ such that

$$d(w(x),w(y)) \le sd(x,y),\ \forall\ x,y \in X. \qquad (1)$$

For $|s|<1$, w is said to be contractive with contrativity s. If w is contractive, it possesses a unique fixed point, x_f, which is also called attractor, so that

$$\lim_{n\to\infty} w^n(x_0) = x_f,\ w(x_f) = x_f, \qquad (2)$$

where is x_0 an initial image. When x and x_{fi} denote an original image and the attractor of w_i, respectively, the optimal transformation can be defined as

$$w_{opt} = \arg\min_{w_i}\left(d(x, \lim_{n\to\infty} w_i^n(x_0))\right). \qquad (3)$$

The number of all the transformations considered in encoding is infinite because the parameters of the transformations are real numbers. However the number is practically finite by virtue of quantization of the parameters. Let S_w denote a set of the quantized transformations. When the attractors of all the elements in S_w are generated, we can obtain an attractor having the minimal distance from an original image. Then the transformation generating the attractor is the optimal one. Such procedure is too heavy to implement in view of computational complexity and hence the transformation is usually found out using the collage theorem, that is,

$$d(x,x_f) \le (1-s)^{-1}\ d(x,w(x)). \qquad (4)$$

III. Optimization of Transformation

In this section, we explain a scheme to obtain more optimal transformation, estimating the optimal attractor.

3.1 PROPOSED SCHEME

Consider a contractive transformation w, for $x,y \in X$, that satisfy

$$d(x, w(y)) \le d(x, y). \quad (5)$$

In Eq. (5), we can observe that one of the transformations satisfying this equation is the optimal transformation for $y = x_{fopt}$. Therefore the transformations are the good candidates for the optimal. In other words, if all the transformations satisfying Eq. (5) for $y = x_{fopt}$ can be obtained, then their attractors are compared with an original image x, and the optimal transformation can be obtained choosing one of the transformations.

Let O_w be a set of the candidates and we examine into the attractors of elements in O_w. From Eq. (5) and the triangular inequality, we have

$$d(x, x_f) \le \frac{1+s}{1-s} d(x, y). \quad (6)$$

Eq. (6) shows that the distance between x and the attractor of an element in O_w is bounded by a contractivity factor s and the distance $d(x,y)$, while the distance in the collage theorem is done by s and $d(x,w_c(x))$ where w_c is a transformation obtained by the collage coding.

Since the optimal transformation is what we hope to find, we should impose constraints on y such that the optimal transformation w_{opt} is an elements of O_w. One of the constraints is clearly $y=x_{fopt}$ because Eq. (5) is satisfied when $w=w_{opt}$ and $y=x_{fopt}$. The constraints satisfying $w_{opt} \in O_w$, for $y \ne x_{fopt}$, will be shown in the next subsection. Anyway, as x_{fopt} is unknown, it should be estimated. Thus our scheme first estimates the optimal attractor before finding the transformation. After the estimation, we obtain elements of O_w satisfying Eq. (5) and then choose the best one among the elements. We can consider that performance of our scheme is determined by accuracy of the estimation, but in the next subsection we will show that the accuracy does not seriously affect the performance of our scheme.

3.2 ESTIMATION OF THE OPTIMAL ATTRACTOR

Consider a special case $y=x_{fopt}$. In this case, the attractors of the elements in O_w are bounded by

$$d(x, x_f) \le \frac{1+s}{1-s} d(x, x_{fopt}). \quad (7)$$

Eq. (7) shows that the elements in O_w can be sub-optimal transformations as $d(x,x_f)$ is bounded by the scaled value of $d(x,x_{fopt})$. However, our scheme has a problem that it is difficult to estimate x_{fopt} accurately.

To solve the problem, we analyze our scheme considering the estimation error. The estimated image of the optimal attractor can be represented as

$$\hat{x}_{fopt} = x_{fopt} + \delta, \quad (8)$$

where δ is the estimation error. We assume that

$$\delta=\beta(x-x_{fopt}), |\beta| \le 1. \quad (9)$$

The assumption in Eq. (9) means that the direction of δ is the same with $(x-x_{fopt})$ and the magnitude of δ is not more than $d(x,x_{fopt})$, i.e., $|\beta| \le 1$. The assumption for magnitude of the error is reasonable because the distance between the original image and the optimal attractor can be roughly predicted by the distance between the original image and the attractor obtained using the conventional collage coding schemes. It is because when an original image is close to the attractor obtained by the collage coding, the original image is also close to the optimal attractor.

Now, we look into the assumption for the direction of the error. If x_{fopt} is compactly distributed on $B(x,r)$, which denotes a ball of radius r, centered at x, the assumption is natural.

From Eq. (9), Eq. (8) is rewritten by

$$\hat{x}_{fopt} = \beta x + (1-\beta) x_{fopt}, |\beta| \le 1. \quad (10)$$

In Eq. (10), if $\beta>0$, $\hat{x}_{fopt}$ is within $B(x,x_{fopt})$, and if $\beta<0$, $\hat{x}_{fopt}$ is out of the ball. In other words, the estimated attractor of $\beta>0$ is closer to x than that of $\beta<0$. For simplicity of notation, putting $\beta=1-\alpha$, Eq. (10) is rewritten by

$$\hat{x}_{fopt} = (1-\alpha)x + \alpha x_{fopt}, 0 \le \alpha \le 2. \quad (11)$$

Substituting $y=\hat{x}_{fopt}$ into Eq. (7), we have

$$d(x, x_f) \le \frac{1+s}{1-s} \cdot \alpha \cdot d(x, x_{fopt}), \quad (12)$$

where we can observe that $d(x,x_f)$ decreases if $\alpha<1$, compared with it in Eq. (7). That is, the smaller α, the more improvement of the picture quality. When $\hat{x}_{fopt}$ is estimated such that it is within the ball $B(x,x_{fopt})$, α can be decreased. However, since a small α gives a small $d(x, \hat{x}_{fopt})$, the number of transformations satisfying Eq. (5) decreases. Extremely, if $\alpha=0$, then $\hat{x}_{fopt}=x$. In this case, there is no transformation satisfying Eq. (5), except x being an attractor.

On the other hand, incorrect estimation of x_{fopt}, i.e., $\delta \ne 0$, doesn't guarantee that $w_{opt} \in O_w$. Thus we modify slightly Eq. (5) such that $w_{opt} \in O_w$, that is,

$$d(x,w(\hat{x}_{fopt})) \le d(x, \hat{x}_{fopt}) + \varepsilon, \quad (13)$$

where ε is a positive number such that $w_{opt} \in O_w$. For the purpose of guaranteeing $w_{opt} \in O_w$, we should select ε that satisfies Eq. (13) for $w= w_{opt}$. Putting max $d(x,w(\hat{x}_{fopt}))=d(x, \hat{x}_{fopt})+\varepsilon$, we have

$$\min \varepsilon=|1-\alpha| d(x,w_{opt}(x)). \quad (14)$$

As Eq. (6) was derived, we obtain

$$d(x, x_f) \le \frac{1+s}{1-s} \cdot \alpha \cdot d(x, x_{fopt}) + \frac{\varepsilon}{1-s}, \quad (15)$$

$$\text{where } \varepsilon = |1-\alpha| d(x, w_{opt}(x)).$$

From the triangular inequality, we obtain the relation that

$(1+s_{opt})d(x,x_{fopt})=\max d(x,w_{opt}(x))$, where s_{opt} is the contractivity factor of w_{opt}, and then Eq. (15) is rewritten by

$$d(x,x_f) \le \left(\alpha(1+s)+|1-\alpha|(1+s_{opt})\right) \cdot \frac{1}{1-s} d(x,x_{fopt}), \quad 0 \le \alpha \le 2. \quad (16)$$

Eq. (16) has the following properties.

(1) For $0\le\alpha\le1$, if $s=s_{opt}$, then the upper bound of Eq. (7) is the same with that of this equation. This is the case that the estimated value is within the ball $B(x,x_{fopt})$. In this case, Eq. (16) means that the estimation biased to an original image x has as good performance as the accurate estimation of the optimal attractor, guaranteeing $w_{opt}\in O_w$. This equation also shows that the accurate estimation is not essential to find more optimal transformation but the estimated value should be close to an original image. Thus the proposed scheme uses such a biased estimation that the estimated image would be close to an original, i.e., *a bias factor*, denoted by α, would be less than 1. Substituting the images obtained by the biased estimation into Eq. (13), we have the transformations satisfying Eq. (13). Then the attractors of the transformations are bounded by Eq. (16). The conventional schemes obtain only one transformation using the collage theorem, but the proposed scheme obtains several transformations satisfying Eq. (13) from such estimated images that $\alpha\le1$, and then chooses the best one.

(2) For $1<\alpha\le2$, the upper bound is looser, compared with that of Eq. (7). This is the case that the estimated value is out of the ball $B(x,x_{fopt})$. We should avoid this case for the purpose of improvement of image quality. However, as x_{fopt} is unknown, α is also unknown. This problem can solved by biased estimation such as the case $0\le\alpha\le1$, i.e., the biased estimation image is obtained by weighted sum of an original image and an estimated image of the optimal attractor.

3.3 Algorithm

In the previous subsection, we showed that the upper bound of the elements in O_w was given by Eq. (16) and the accurate estimation was not necessary because the biased estimation was as good as accurate estimation if $0\le\alpha\le1$. Thus we can consider the filtered images of an original image as an estimated images of the optimal attractor because the images are close to an original image. We experimentally obtained that the method by filtering has low performance although the method is superior to the conventional methods. It may be because $\hat{x}_{fopt}$ don't have the structure of the attractor, i.e., fractal structure. So, in this subsection, we propose an estimation method using attractors.

The procedure is:

(1) Obtain a transformation using the conventional collage coding scheme, which corresponds to the case $\alpha=0$, i.e., $x=\hat{x}_{fopt}$, and then put the transformation to w_0. That is, w_0 is obtained by

$$w_0=\arg\min_{w_i} d(x,w_i(x)). \quad (17)$$

As it is always true that $d(x,x_{f0})\ge d(x,x_{fopt})$, where x_{f0} is the attractor of w_0, if $\hat{x}_{fopt}=x_{f0}$, then $\alpha\ge1$, i.e., if x_{f0} is used as an estimated image, the upper bound increases as shown in Eq. (16).

(2) To solve the problem that $\alpha>1$ in step 1, we set a biased estimation image $\hat{x}^*_{fopt}$ to weighted sum of x and x_{f0}.

$$\hat{x}^*_{fopt} = (1-\rho)x+\rho x_{f0}, \quad 0\le\rho\le1, \quad (18)$$

where $(1-\rho)x$ is added to guarantee that the bias factor of the biased estimation image has a small value. Let α^* and α denote the bias factors of $\hat{x}^*_{fopt}$ and x_{f0}, respectively. Putting $x_{f0}=(1-\alpha)x+\alpha x_{fopt}$ as shown in Eq. (11), we have

$$\hat{x}^*_{fopt} = (1-\rho\alpha)x+\rho\alpha x_{fopt}$$
$$= (1-\alpha^*)x+\alpha^* x_{fopt}, \quad 0\le\rho\le1,\ 0\le\alpha\le2. \quad (19)$$

Eq. (19) shows that $\alpha^*=\rho\alpha$, where α is unknown value determined by accuracy of estimation but ρ is a controllable value given by designer. Thus α^* can be controlled by ρ. If ρ is constrained to be in $0\le\rho\le1/2$, it is guaranteed that $\alpha^*\le1$. If α is slightly larger than 1, i.e., x_{f0} is not distant from x_{fopt}, then ρ can be more than 1/2.

After $\hat{x}^*_{fopt}$ is obtained in Eq. (19), find a transformation w that minimize $d(x,w(\hat{x}^*_{fopt}))$. Then put the transformation to w_1.

(3) For generalization of notation, let $y_1 = \hat{x}^*_{fopt}$, where $\hat{x}^*_{fopt}$ was obtained in step 2, and let x_{f1} be the attractor of w_1. Then we use a new estimated image

$$y_2=(1-\rho)y_1+\rho x_{f1}, \quad 0\le\rho\le1. \quad (20)$$

After y_2 is obtained in Eq. (20), find a transformation w that minimizes $d(x,w(y_2))$. Then put the transformation to w_2. By the similar procedure, y_n is given by

$$y_n=(1-\rho)y_{n-1}+\rho x_{f(n-1)}, \quad 0\le\rho\le1, \quad (21)$$

where $x_{f(n-1)}$ is the attractor of w_{n-1}. After y_n is obtained in Eq. (21), find a transformation w that minimizes $d(x,w(y_n))$.

(4) Obtain w_i and x_{fi}, for $0\le i\le N$. Then we decide a transformation w^* to be encoded such that

$$w^* = \arg\min_{w_i} d(x,x_{fi}),\ for\ 0\le i\le N. \quad (22)$$

The procedure of the estimation is shown in Figure 1.

IV. Experimental Results

To evaluate the performance of the proposed schemes, computer simulations were performed for a simple case (C1) and general case (C2). C1 is the case that shuffling and searching for domain blocks are not considered, i.e., domain blocks are prefixed by range blocks. After the optimal transformation is obtained, the proposed methods and the conventional method are evaluated. As we cannot unfortunately obtain the optimal transformation for C2, the proposed schemes are evaluated, compared with the conventional method. Let PM1 and PM2 denote the proposed methods using the filtering and the attractors, respectively.

First we tested each method for C1. 256x256 Lena and 352x288 Akiyo are used for the tests. 16x16 domain blocks and 8x8 range blocks are used, 8 bits are allocated to the translational terms of luminance, and contractivity factors are in [0.05, 1.0]. To evaluate the performance of each method, the contractivity factors are quantized to step size 0.05 so that the number of the possible transformations may be enough. The coding results for C1 are shown in Table 1. The results of the optimal coding for two images are shown in the first row where mean square error (MSE) for Lena image is 383.84 and for Akiyo is 130.67. We can observe that PM1 and PM2 are superior to the conventional method. Particularly, PM2 is very close to the optimal. PM1 used simply 2x2 averaging filter, which was the same with the spatial contraction operator used widely in the conventional fractal coding schemes. The factor ρ for the bias estimation in PM2 was experimentally determined. For $0.3 \le \rho \le 0.7$, we obtained good performance regardless of the images. Thus we chose $\rho=0.5$. And we used $N=8$ in step 4 of PM2. As Akiyo image is smooth, the conventional method, PM1, and PM2 have similar performance but PM2 is still close to the optimal coding.

Now each method is evaluated for C2. Two kinds of 8x8 and 16x16 range blocks are used. Unlike C1, 8 shuffle transformations are applied, and searching ranges for domain blocks are set to a quarter of an entire image for 8x8 range blocks and an entire image for 16x16 range blocks. The searching step for domain blocks is set to be equal to the size of range block, i.e., 8 for 8×8 range block and 16 for 16×16 range block. The contractivity factors and the translational terms follow C1. The results by each method applied to 256x256 Lena image are shown in Table 2. PM2 has about 0.3dB gain for Lena image, compared with the conventional method. From the simulation results, we can conclude that PM2 has the best performance. Why PM2 is superior to PM1 is because we guess the structure of the estimated images obtained by PM2 is more similar to the structure of an attractor than that of PM1.

V. Conclusions

We proposed a new fractal coding scheme to obtain more optimal transformation by estimation of the optimal attractor. Unlike the conventional methods using the collage theorem, our schemes first obtain the good candidates of the optimal transformation and then we determine the best one among them. The candidates are the transformations to minimize the distances between an original image and the estimated images of the optimal attractor.

We evaluated the performance of our schemes for a simple case and general cases. Resultantly, we obtained that PM2 was very close to the optimal method for a simple case and also had about 0.3dB gain for general cases, compared with conventional method.

The choice of $\hat{x}_{fopt}$'s is very important because it affects the performance of coder. Thus we expect that good choice of $\hat{x}_{fopt}$'s gives improvement of the performance.

References

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Table 1 Comparison of each method for a simple case

methods	MSE(Lena)	MSE(Akiyo)
optimal coding	383.84	130.67
PM1	396.40	132.00
PM2	385.64	131.60
collage coding	405.51	138.43

Table 2 Comparison each method for general case (256x256 Lena, ρ=0.5)

methods	MSE(PSNR) (8x8 block)	MSE(PSNR) (16x16 block)
collage coding	169.86(25.83)	361.48(22.55)
PM1	167.53(25.89)	350.00(22.69)
PM2	159.25(26.11)	335.80(22.87)

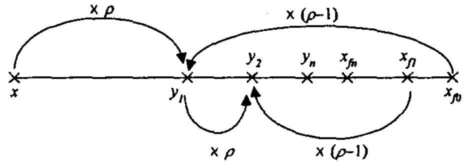

Figure 1 Estimation by attractors