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Free Vibration Analysis of Solid and Annular Circular Membranes with Continuously Varying Density Using The Differential Transformation Method

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Key Words : Annular Membrane, Solid Membrane, Differential Transformation Metheod

ABSTRACT

This paper presents the application of the technique of differential transformation of free vibration of membrane. Numerical calculations are carried out and compared with previously published results. The results obtained by the present method agree very well with those reported in the previous works. The present analysis shows the usefulness and validity of differential transformation in solving a solid and annular circular membranes problem of the responses of the free vibration.

1. Introduction

The problem of determining the natural frequencies of membranes is important in the design of many engineering devices. These include microphones, louds- peakers, pumps, compressors and pressure regulators, antennae for space communications, etc. A general review of the dynamic aspects of membranes can be found in review papers by Mazumdar [1]. Recently, there have been many studies of the non-homogeneous membrane whose non-homogeneity is piecewise continuous [2-5]. However, there are few studies on the non-homogeneous membrane whose non-homogeneity is continuous in all domains. Masad [6] solved the problem mentioned above by the finite difference method and the perturbation method. Laura [7] solved the same problem by the optimized Galerkin- Kantrovitch approach and the differential quadrature method. In this study, the differential transformation method is introduced to solve the above referenced problems. The concept of differential transforms was first proposed by Zhou [8] in 1986 and was applied to solve linear and non-linear initial value problems in electric circuit analysis. Using differential transforms, Chen and Ho [9] proposed a method to solve eigenvalue problems.

In this paper, the free vibration problems of a solid and circular annular membranes with continuouly varying density are considered. Using the differential transform technique, natural frequenies of solid and annular circular membranes can be obtained. Finally, the fundamental natural frequencies of a solid and annular membranes are investigated to illustrate the accuracy and efficiency of the

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present method.

2. Theory

The current work deals with two situations:
(1) a solid circular membrane of radius R
(2) an annular membrane of outer radius R and inner radius R_0 .
The membrane density is assumed to be of the form

$$\rho(r) = \rho_0 f(r)$$

The governing differential equation for the displacement $W(r)$ for the axisymmetric vibrational modes in the case of a solid circular membrane or an annular circular membrane of outer radius R and inner radius R_0 is

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} + \Omega^2 f(r) W(r) = 0$$
$$0 \leqq r_0 \leqq r \leqq 1 \qquad (1)$$

where

$$r_0 = \frac{R_0}{R} \quad , \quad f(r) = \sum_{n=0}^{\infty} f_n r^n \ ,$$
$$\Omega = wR\sqrt{\rho_0/S}$$

and S is the tension per unit length[10]. The coefficients f_n are not restricted but $\sum_{n=0}^{\infty} f_n$ must be a convergent series ($\sum_{n=0}^{\infty} f_n < \infty$).
In order to handle the problem of annular and solid circular membranes, it is convenient to introduce the new variable

$$r^* = 1 - r \qquad (2)$$

and equation (1) becomes

$$\frac{d^2W}{dr^*} + \frac{1}{(r^*-1)}\frac{dW}{dr^*} + \Omega^2 W(r^*) f(r^*) = 0$$
$$0 \leqq r^* \leqq 1 - r_0 \qquad (3)$$

Equation (3) can be rewritten as follows

$$(r^*-1)\frac{d^2W}{dr^*} + \frac{dW}{dr^*} + (r^*-1)\,\Omega^2 W(r^*) f(r^*) = 0$$
$$0 \leqq r^* \leqq 1 - r_0 \qquad (4)$$

where

$$f(r^*) = \sum_{n=0}^{\infty} f_{n2}(r^*)^n = \sum_{n=0}^{\infty} f_{n1} r^n = f(r) \qquad (5)$$

The boundary conditions for solid circular membrane are follow

$$\frac{dW}{dr}(r=0) = -\frac{dW}{dr^*}(r^*=1) = 0 \qquad (6)$$
$$W(r=1) = W(r^*=0) = 0 \qquad (7)$$

and the boundary condition for the annular circular membrane are

$$W(r=r_0) = W(r^* = 1 - r_0) = 0 \qquad (8)$$
$$W(r=1) = W(r^*=0) = 0 \qquad (9)$$

3. Differential Transformation

In order to solve the ordinary differential equation (3) by differential transformation, its basic theory is stated briefly.

Let $y(x)$ be an analytic in domain D and $x=0$ be a point in D. Then there exists precisely one power series with center at $x=0$ which represents $y(x)$ this series, the Maclaurin series of the function $y(x)$, is in the form

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \qquad (10)$$

for $\forall x \subset D$

If we define differential transformation of function $y(x)$ as follows

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \qquad (11)$$

and substitute equation (11) into equation (10), equation (10) becomes

$$y(x)=\sum_{k=0}^{\infty} x^k Y(k) \qquad (12)$$

$Y(k)$ is the differential transformation (T-function) for the original function $y(x)$, and equation (12) is the differential inverse transformation of $Y(k)$ [8,9].

From the above definition of the Differential Transformation of the function. We can derive the rules of transformational operations: some of these, which are useful in the following analysis, are as follows:

Orginal function	T - function
$w(x)=y(x)\pm z(x)$	$W(k)=Y(k)\pm Z(k)$
$z(x)=\lambda y(x)$	$Z(k)=\lambda Y(k)$ λ, constant
$z(x)=\frac{dy(x)}{dx}$	$Z(k)=(k+1)Y(k+1)$
$w(x)=\frac{d^2y(x)}{dx^2}$	$W(k)=(k+1)(k+2)Y(k+2)$
$w(x)=y(x)z(x)$	$W(k)=\sum_{l=0}^{k} Y(l)Z(k-l)$
$w(x)=x^m$	$W(k)=\delta(k-m)$ at 1 $k=m$, 0 $k\neq m$

In actual applications, the function y(x) may be expressed by a finite series and equation (12) can be written as

$$y(x)=\sum_{k=0}^{n} x^k Y(k) \qquad (13)$$

Equation (13) implies that $\sum_{k=n+1}^{\infty} x^k Y(k)$ is neglected. Generally, n is decided by the desired convergence of the natural frequency.

4. Application of Differential Transformation to Solid and Annular Circular Membranes

Taking differential transformation of equation (4) and using the transformational operations mentioned above, we obtain

$$\begin{aligned}
&\sum_{i=0}^{k}\delta(i-1)(k+1-i)(k+2-i)W(k+2-i)\\
&-(k+1)(k+2)W(k+2)\\
&+(k+1)W(k+1)\\
&+\Omega^2\sum_{i=0}^{k}W(k-i)\sum_{n=0}^{i}\delta(n-1)f(i-n)\\
&-\Omega^2\sum_{i=0}^{k}W(k-i)f(i)=0 \qquad (14)
\end{aligned}$$

Where $W(k)$ and $f(k)$ are T-functions of $W(r^*)$ and $f(r^*)$ respectively. In order to analyze the membranes, the boundary conditions should be transformed. Using equation(10), the boundary condition equations (6-9) become

$$W(0)=0 \quad , \quad \sum_{k=0}^{n} k\times 1^{(k-1)}W(k)=0 \qquad (15,16)$$

for the solid circular membrane,

$$W(0)=0 \quad , \quad \sum_{k=0}^{n}(r^*)^k W(k)=0 \qquad (17,18)$$

for the annular circular membrane

5. Numerical Results and Discussions

In order to obtain the natural frequencies of the membrane, we should take advantage of transformed equation (14) and the two corresponding boundary conditions among equations (15-18). These can be represented in matrix form as

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} & a_{1,k+1} \\ a_{2,1} & a_{2,2} & & a_{2,k} & a_{2,k+1} \\ \cdot & \cdot & & \cdot & \cdot \\ a_{k+1,1} & a_{k+1,2} & & a_{k+1,k} & a_{k+1,k+1} \end{bmatrix} \begin{bmatrix} W(0) \\ W(1) \\ \cdot \\ W(k) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{19}$$

A non-trivial solution exists when the determinant of the coefficient matrix vanishes. This condition leads to the following frequency equation:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & a_{1,k} & a_{1,k+1} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & \cdots & a_{2,k} & a_{2,k+1} \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot & \cdot \\ a_{k+1,1} & a_{k+1,2} & \cdots & \cdots & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{vmatrix} = 0 \tag{20}$$

The solution of equation (20) yields the desired frequency parameter Ω of membranes

Example 1

Consider the first case that

$$f = 1 + \alpha r^2 \tag{21}$$

which has been studied in reference [11]. In Table 1, 2 and 3 values of Ω corresponding to the fundamental frequency coefficients are given for the case $\alpha = 0.5, 1, 1.5$. The agreement with references [11] is excellent. The same conclusion is reached for other α values.

Table.1

Fundamental natural frequency Ω_1, case with $f(r) = 1 + 0.5r^2; r_0 = 0$ indicates solid circular membrane

	FEM[11]	DQM[11]	DT
r_0	Ω_1	Ω_1	Ω_1
0	2.2820	2.2819	2.2819
0.1	3.0773	3.0787	3.0735
0.2	3.4998	3.4974	3.4969
0.3	3.9943	3.9943	3.9943
0.4	4.6320	4.6320	4.6321
0.5	5.5070	5.5070	5.5071
0.6	6.8095	6.8050	6.8050
0.7	8.9606	8.9547	8.9547
0.8	13.2482	13.2394	13.2394

Table.2

Fundamental natural frequency Ω_1, case with $f(r) = 1 + r^2; r_0 = 0$ indicates solid circular membrane

	FEM[11]	DQM[11]	DT
r_0	Ω_1	Ω_1	Ω_1
0	2.1738	2.1735	2.1735
0.1	2.8787	2.8797	2.8753
0.2	3.2457	3.2435	3.2430
0.3	3.6757	3.6730	3.6730
0.4	4.2262	4.2233	4.2233
0.5	4.9815	4.9781	4.9781
0.6	6.1025	6.0984	6.0984
0.7	7.9610	7.9557	7.9557
0.8	11.6693	11.6616	11.6616

Table.3

Fundamental natural frequency Ω_1, case with $f(r) = 1 + 1.5r^2$; $r_0 = 0$ indicates solid circular membrane

	FEM[11]	DQM[11]	DT
r_0	Ω_1	Ω_1	Ω_1
0	2.0779	2.0778	2.0778
0.1	2.7122	2.7130	2.7092
0.2	3.0381	3.0361	3.0358
0.3	3.4195	3.4171	3.4171
0.4	3.9081	3.9053	3.9054
0.5	4.5793	4.5762	4.5762
0.6	5.5773	5.5737	5.5736
0.7	7.2348	7.2300	7.2300
0.8	10.5467	10.5397	10.5397

Example 2

Consider the case with

$$f(r) = 1 + \alpha r + \beta r^2 + \gamma r^3 + \delta r^4 \quad (22)$$

where α, β, γ and δ are constants. The expression for f in terms of r^* coordinates becomes

$$f(r^*) = 1 + (\alpha+\beta+\gamma+\delta) - (\alpha+2\beta+3\gamma+4\delta)r^* + (\beta+3\gamma+6\delta)r^{*2} - (\gamma+4\delta)r^{*3} + \delta r^{*4} \quad (23)$$

In Table 4, the first three natural frequency coefficient values are given for the case of $\alpha=1,\ \beta=1,\ \gamma=0,\ \delta=0$

In Table 5, the first three natural frequency coefficient values are given for the case with $\alpha=1,\ \beta=1,\ \gamma=1,\ \delta=0$

In Table 6, the first three natural frequency coefficient values are given for the case of $\alpha=1,\ \beta=1,\ \gamma=1,\ \delta=1$

Table.4

The first three natural frequencies, case with $f(r) = 1 + r + r^2$; $r_0 = 0$ indicates solid circular membrane

r_0	D.T. Method		
	Ω_1	Ω_2	Ω_3
0	1.8655	4.1704	6.5028
0.1	2.4155	4.9942	7.5594
0.2	2.7045	5.5248	8.3297
0.3	3.0456	6.1766	9.2934
0.4	3.4859	7.0351	10.5725
0.5	4.0941	8.2345	12.3657
0.6	5.0015	10.0359	15.0635
0.7	6.5114	13.0450	19.5737
0.8	9.5314	19.0761	28.6178
0.9	18.5963	37.1986	55.7996

Table.5

The first three natural frequencies, case with $f(r) = 1 + r + r^2 + r^3$; $r_0 = 0$ indicates solid circular membrane

r_0	D.T. METHOD		
	Ω_1	Ω_2	Ω_3
0	1.7910	3.9557	6.1575
0.1	2.2850	4.7064	7.1191
0.2	2.5393	5.1763	7.8013
0.3	2.8373	5.7478	8.6464
0.4	3.2203	6.4959	9.7612
0.5	3.7479	7.5370	11.3179
0.6	4.5338	9.0975	13.6550
0.7	5.8408	11.7022	17.5591
0.8	8.4549	16.9224	25.3872
0.9	16.3035	32.6129	48.9210

Table.6

The first three natural frequencies, case with $f(r) = 1 + r + r^2 + r^3 + r^4$; $r_0 = 0$ indicates solid circular membrane

r_0	D.T. METHOD		
	Ω_1	Ω_2	Ω_3
0	1.7465	3.8158	5.5299
0.1	2.2055	4.5216	6.8348
0.2	2.4378	4.9541	7.4624
0.3	2.7081	5.4752	8.2335
0.4	3.0537	6.1526	9.2433
0.5	3.5278	7.0902	10.6458
0.6	4.2322	8.4905	12.7432
0.7	5.4020	10.8226	16.2391
0.8	7.7401	15.4921	23.2414
0.9	14.7594	29.5245	44.2883

6. Conclusions

In this paper the technique of differential transform has been instroduced for the vibration problem of solid and annular circular membranes , a general quasi−

analytical model based on the differential transform method is described so as to handle vibrations of solid and annular circular membranes with continuously varying density. The method given in this work serves as an extension to previous analytical works as it can be used to handle any density variation which can be represented as an infinite series in the radial coordinate. The results of natural frequencies claculated from the differential transformation solutions are compared with those by finite element mothod and differential quadrature method. The calculated results show quite clearly that these are fast converging series . and further, the calculated results from the differential tranformation solutions are very high accuracy with respect to finite element mothod and differential quadrature method.

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