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V노치 또는 예리한 균열을 가지는 Mindlin 직사각형 평판의 휨 진동해석

Flexural Vibration Analysis of Mindlin Rectangular Plates Having V-notches or Sharp Cracks

김 주 우* 정 의 영** 김 승 현**
Kim, Joo-Woo Jung Eui-Young Kim, Seung-Hyun

ABSTRACT

This paper provides the first known flexural vibration data for thick (Mindlin) rectangular plates having V-notches. The V-notch has bending moment and shear force singularities at its sharp corner due to the transverse vibratory bending motion. Based upon Mindlin plate theory, in which transverse shear deformation and rotary inertia effects are considered, the Ritz procedure is employed with a hybrid set of admissible functions assumed for the rotational and transverse vibratory displacements. This set includes: (1) a mathematically complete set of admissible algebraic-trigonometric polynomials which guarantee convergence to exact frequencies as sufficient terms are retained; and (2) an admissible set of Mindlin *corner functions* which account for the bending moment and shear force singularities at the sharp corner of the V-notch. Extensive convergence studies demonstrate the necessity of adding the Mindlin corner functions to achieve accurate frequencies for rectangular plates having sharp V-notches.

1. INTRODUCTION

The problem of free vibration of complete rectangular, thin and thick plates has attracted the attention of many researchers. However, the scope of previous work done for vibrations of completely free thick rectangular plates having V-notches or cracks is scarce. First order shear deformation theories by Reissner(1) and Mindlin(2), that include the effect of shear deformation and rotary inertia, have also been used in the vibration analysis of moderately thick rectangular or annular sectorial plates. Huang *et al.*(3) provided exact analytical solutions for the free vibrations of Mindlin sectorial plates with simply supported radial edges that formed reentrant corners having unbounded bending stresses, and arbitrary circumferential edge conditions. Recently, based upon the Mindlin plate theory, it has been demonstrated by Kim(4)

* 정회원 · 세명대학교 건축공학과 조교수
** 세명대학교 건축공학과 석사과정

that use of the corner functions is essential to obtaining accurate frequencies of completely free circular plates with V-notches.

This paper provides a new comprehensive database of accurate frequency solutions for completely free Mindlin rectangular plates having V-notches (see Fig. 1). The relative notch depth is defined as $(a-c)/a$ and the notch angle is defined as $(360^\circ-\alpha)$. For a very small notch angle, the notch may be regarded as a sharp radial crack.

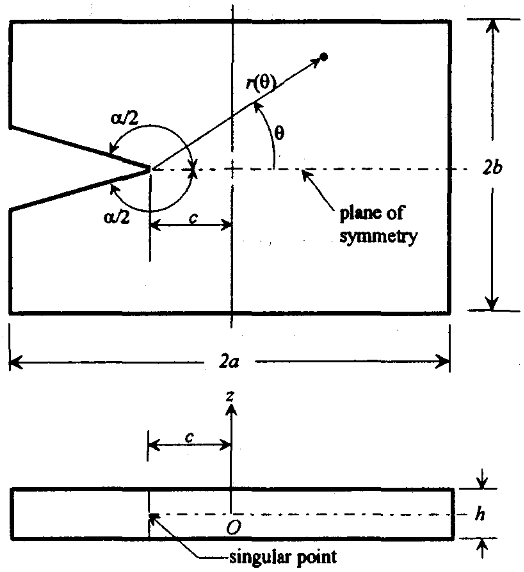

Fig. 1 Completely free Mindlin rectangular plate with a V-notch

A Ritz procedure is employed which incorporates a complete set of admissible algebraic-trigonometric polynomials in conjunction with an admissible set of Mindlin corner functions. These corner functions model the singular vibratory moments and shear forces which simultaneously exist at the vertices of corner angles (α) exceeding 180°. The first set guarantees convergence to exact frequencies when sufficient terms are retained. The second set substantially accelerates frequency convergence, which is demonstrated using numerical studies. Reported in this paper is an accurate database of non-dimensional frequencies for a wide spectrum of corner angles and relative notch depths. Comparison studies are also performed with existing results[5] of classically thin rectangular plates having V-notches.

2. METHODOLOGY

Consider in Fig. 1 a completely free, thick notched rectangular plate having length $2a$, width $2b$ and thickness h with polar coordinates (r,θ) at the middle surface. The vibratory rotations and a transverse displacement are assumed in terms of polar coordinates (r,θ) at the middle surface as

$$\phi_r(r,\theta,t) = \Phi_r(r,\theta)\sin wt, \quad \phi_\theta(r,\theta,t) = \Phi_\theta(r,\theta)\sin wt, \quad w_z(r,\theta,t) = W_z(r,\theta)\sin wt, \tag{1}$$

where t is time and w is the circular frequency of vibration. In using the Ritz method, one requires the maximum values of strain energy and kinetic energy, which occur during a cycle of vibratory motion.

The maximum strain energy in the Mindlin plate due to bending during a vibratory cycle is

$$\begin{aligned} V_{\max} = \frac{1}{2}\iint \Bigg\{ D\Bigg[\left(\frac{\partial\Phi_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\Phi_r + \frac{\partial\Phi_\theta}{\partial\theta}\right)^2 + \frac{2\nu}{r}\frac{\partial\Phi_r}{\partial r}\left(\Phi_r + \frac{\partial\Phi_\theta}{\partial\theta}\right) \\ + \frac{(1-\nu)}{2}\left(\frac{1}{r}\frac{\partial\Phi_r}{\partial\theta} - \frac{\Phi_\theta}{r} + \frac{\partial\Phi_r}{\partial r}\right)^2\Bigg] \\ + \kappa^2 Gh\Bigg[\left(\Phi_r + \frac{\partial W_z}{\partial r}\right)^2 + \left(\Phi_\theta + \frac{1}{r}\frac{\partial W_z}{\partial\theta}\right)^2\Bigg]\Bigg\} r\,dr\,d\theta, \end{aligned} \tag{2}$$

where $D = Eh^3/12(1-\nu)$ is flexural rigidity, $G = E/2(1+\nu)$ is shear modulus, E is Young's modulus, ν is Poisson's ratio, and κ^2 is shear correction factor. The maximum kinetic energy is

$$T_{\max} = \frac{\rho w^2}{2}\iint \left[h W_z^2 + \frac{h^3}{12}(\Phi_r^2 + \Phi_\theta^2)\right] r\,dr\,d\theta, \tag{3}$$

in which ρ is the mass per unit area of the plate. The area integral given in Eq. (3) may be performed for rectangular plate using the following values of $r(\theta)$:

$$r(\theta) = \frac{b}{\sin\theta} \quad (\theta_1 \leqq \theta < \theta_2), \qquad r(\theta) = -\frac{a-c}{\cos\theta} \quad (\theta_2 \leqq \theta < \theta_3), \tag{4a}$$

$$r(\theta) = -\frac{b}{\sin\theta} \quad (\theta_3 \leqq \theta < \theta_4), \qquad r(\theta) = -\frac{a+c}{\cos\theta} \quad (\theta_4 \leqq \theta < \theta_1), \tag{4b}$$

where

$$\theta_1 = \tan^{-1}\left(\frac{b}{a+c}\right), \qquad \theta_2 = -\tan^{-1}\left(\frac{b}{a-c}\right),$$

$$\theta_3 = \tan^{-1}\left(\frac{b}{a-c}\right), \qquad \theta_4 = -\tan^{-1}\left(\frac{b}{a+c}\right).$$

In the present Ritz approach, displacement trial functions are assumed as the sum of two finite sets:

$$\Phi_r = \Phi_r^p + \Phi_r^c, \qquad \Phi_\theta = \Phi_\theta^p + \Phi_\theta^c, \qquad W_z = W_z^p + W_z^c \tag{5}$$

where Φ_r^p, Φ_θ^p, and W_z^p are algebraic-trigonometric polynomials and Φ_r^c, Φ_θ^c, and W_z^c are Mindlin corner functions. The admissible polynomials are written as

$$\Phi_r^p = \sum_{m=2,4}^{M_1}\sum_{n=0,2,4}^{m} A_{mn} r^{m-1}\cos n\theta + \sum_{m=1,3,5}^{M_2}\sum_{n=1,3,5}^{m} A_{mn} r^{m-1}\cos n\theta \tag{6a}$$

$$\Phi_\theta^p = \sum_{m=2,4}^{M_3}\sum_{n=2,4}^{m} B_{mn} r^{m-1}\sin n\theta + \sum_{m=1,3,5}^{M_4}\sum_{n=1,3,5}^{m} B_{mn} r^{m-1}\sin n\theta \tag{6b}$$

$$W_z^p = \sum_{m=0,2,4}^{M_5}\sum_{n=0,2,4}^{m} C_{mn} r^{m-1}\cos n\theta + \sum_{m=1,3,5}^{M_6}\sum_{n=1,3,5}^{m} C_{mn} r^{m-1}\cos n\theta \tag{6c}$$

for the symmetric vibration modes, and

$$\Phi_r^p = \sum_{m=2,4}^{M_1}\sum_{n=2,4}^{m} D_{mn} r^{m-1}\sin n\theta + \sum_{m=1,3,5}^{M_2}\sum_{n=1,3,5}^{m} D_{mn} r^{m-1}\sin n\theta \tag{7a}$$

$$\Phi_\theta^p = \sum_{m=2,4}^{M_3}\sum_{n=0,2,4}^{m} E_{mn} r^{m-1}\cos n\theta + \sum_{m=1,3,5}^{M_4}\sum_{n=1,3,5}^{m} E_{mn} r^{m-1}\cos n\theta \tag{7b}$$

$$W_z^p = \sum_{m=0,2,4}^{M_5}\sum_{n=0,2,4}^{m} F_{mn} r^{m-1}\sin n\theta + \sum_{m=1,3,5}^{M_6}\sum_{n=1,3,5}^{m} F_{mn} r^{m-1}\sin n\theta \tag{7c}$$

for the antisymmetric modes. In Eqs. (6) and (7), A_{mn}-F_{mn} are arbitrary coefficients, and the values of m and n have been specially chosen to eliminate those terms which yield undesirable singularities at $r = 0$ and yet preserve the mathematical completeness of the resulting series, as sufficient terms are retained.

The displacement polynomial Eqs. (6) and (7) should, in principle, yield accurate frequencies. However, the number of terms may be computationally prohibitive. This problem is alleviated by augmentation of the displacement polynomial trial set with admissible *corner functions*, which introduce the proper singular vibratory moments and shear forces at the vertex of the V-notch (Fig. 1). The set of corner functions is taken as, for the symmetric modes:

$$\Phi_r^c = \sum_{k=1}^{K_1} G_k r^{\lambda_k}[\zeta_k\cos(\lambda_k+1)\theta - \gamma_k\cos(\lambda_k-1)\theta] \tag{8a}$$

$$\Phi_\theta^c = \sum_{k=1}^{K_2} G_k r^{\lambda_k}[-\zeta_k\sin(\lambda_k+1)\theta + \gamma_k\sin(\lambda_k-1)\theta] \tag{8b}$$

$$W_z^c = \sum_{k=1}^{K_3} H_k r^{\hat{\lambda}_k+1}\cos(\hat{\lambda}_k+1)\theta \tag{8c}$$

with

$$\zeta_k = \frac{(\gamma_k + 1)(\lambda_k - 1)\sin(\lambda_k - 1)\alpha/2}{2\lambda_k \sin(\lambda_k - 1)\alpha/2}, \text{ where } \gamma_k = \frac{\lambda_k(1+\nu) - 3 + \nu}{\lambda_k(1+\nu) + 3 - \nu} \tag{8d}$$

In Eqs. (8), the λ_k and $\hat{\lambda}_k$ are the roots of the characteristic equations

$$\sin\lambda_k\alpha = -\lambda_k \sin\alpha, \qquad \sin(\hat{\lambda}_k + 1)\alpha/2 = 0 \tag{9}$$

Similarly, the corner functions used for antisymmetric modes are analogous to those defined for the symmetric ones in Eqs. (8), except the cosine functions are changed to sine functions, or vice versa, and the corresponding characteristic equations are

$$\sin\lambda_k\alpha = \lambda_k \sin\alpha, \qquad \cos(\hat{\lambda}_k + 1)\alpha/2 = 0 \tag{10}$$

Some of the λ_k obtained from Eqs. (9) and (10) may be complex numbers, and thus result in complex corner functions. In such cases, both the real and imaginary parts are used as independent functions in the present Ritz procedure.

The free vibration problem is solved by substituting Eqs. (5)-(8) into Eqs. (2) and (3) and employing the frequency equations of the Ritz method. For the symmetric modes, for example, these are:

$$\frac{\partial(V_{\max} - T_{\max})}{\partial A_{mn}} = 0, \qquad \frac{\partial(V_{\max} - T_{\max})}{\partial B_{mn}} = 0, \qquad \frac{\partial(V_{\max} - T_{\max})}{\partial C_{mn}} = 0 \tag{11a}$$

$$\frac{\partial(V_{\max} - T_{\max})}{\partial G_k} = 0, \qquad \frac{\partial(V_{\max} - T_{\max})}{\partial H_k} = 0 \tag{11b}$$

This results in a set of linear homogeneous algebraic equations involving the coefficients A_{mn}, B_{mn}, C_{mn}, G_k, and H_k. The vanishing determinant of these equations yields a set of eigenvalues (natural frequencies), expressed in terms of the non-dimensional frequency parameter, $wa^2\sqrt{\rho h/D}$ commonly used in the plate vibration literature.

3. CONVERGENCE STUDIES

Having outlined the Ritz procedure employed in the preceding sections, it is now appropriate to address the important question of the convergence accuracy of frequencies as sufficient numbers of algebraic-trigonometric polynomials and Mindlin corner functions are retained. Depicted in Fig. 2 are four representative configurations, which are examined in this work. All of the frequency data discussed in the present section are for materials having shear correction factor κ^2 equal to $\pi^2/12$ and Poisson's ratio ν equal to 0.3.

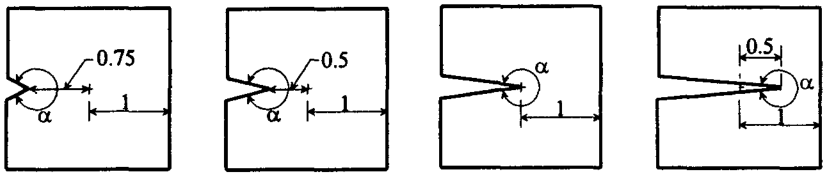

Fig. 2 V-notches with various depths

Table 1 Convergence of frequency parameters $wa^2\sqrt{\rho h/D}$ for a completely free Mindlin square plate having V-notch (a/b = 1, α=355°, a/h = 20, c/a = 0)

Mode no. (symmetry class)*	No. of Corner Functions	Solution size of polynomials				
		18	19	20	21	22
1 (A)	0	3.294	3.293	3.292	3.291	3.291
	5	2.150	2.140	2.130	2.121	2.113
	10	2.122	2.114	2.109	2.102	2.098
	20	2.091	2.087	2.084	2.081	2.078
	25	2.087	2.083	2.080	2.077	2.074
2 (S)	0	4.894	4.893	4.892	4.891	4.890
	5	3.765	3.761	3.757	3.753	3.751
	10	3.731	3.729	3.727	3.725	3.724
	20	3.728	3.726	3.725	3.723	3.722
	25	3.727	3.725	3.724	3.722	3.720
3 (S)	0	5.982	5.980	5.979	5.977	5.975
	5	5.470	5.471	5.470	5.469	5.469
	10	5.465	5.465	5.464	5.464	5.463
	20	5.464	5.464	5.464	5.463	5.463
	25	5.463	5.463	5.462	5.462	5.461
4 (A)	0	8.485	8.480	8.477	8.473	8.468
	5	5.921	5.907	5.892	5.883	5.871
	10	5.880	5.871	5.863	5.856	5.849
	20	5.846	5.841	5.835	5.831	5.827
	25	5.839	5.834	5.828	5.823	5.820

* (S) symmetric mode; (A) antisymmetric mode

Summarized in Table 1 is the first four non-dimensional frequency parameter $wa^2\sqrt{\rho h/D}$ for a completely free, notched square plate with vertex angle (α) of 355° and with thickness ratio (a/h) of 20. This example is appropriately described as a Mindlin square plate with a sharp notch or radial crack. It should be noted that there are three rigid body modes in the vibration of the completely free Mindlin sectorial plate which are not shown in the table. It can be seen in Table 1 that the fundamental frequency mode is an antisymmetric one. An upper bound convergence of frequencies to an inaccurate value of 3.291 is apparent, as the sizes of polynomial series are increased with no Mindlin corner functions. Adding five corner functions improves the convergence rate significantly. Indeed, the trial set consisting of the first five corner functions along with 18 polynomial solutions yields an upper bound frequency value of 2.150. An examination of the next three rows of data reveals that an accurate value to four significant figures is 2.074.

In Table 2, a convergence study of $wa^2\sqrt{\rho h/D}$ for a Mindlin rectangular plate (a/b = 2) is shown for the case of α = 355°. It is seen in Table 2 that as the solution sizes of hybrid set of polynomial and corner functions increase, a slight deterioration in the overall convergence occurs due to onset on matrix ill-conditioning and numerical round-off errors. Despite the ill-conditioning, a convergence to three or four significant figures is essentially achieved for the first four frequencies.

Table 2 Convergence of frequency parameters $wa^2\sqrt{\rho h/D}$ for a completely free Mindlin rectangular plate (a/b = 2, α = 355°, a/h = 20, c/a = 0)

Mode no. (symmetry class)	No. of Corner Functions	Solution size of polynomials				
		11	12	13	14	15
1 (A)	0	1.613	1.610	1.609	1.604	1.600
	5	0.811	0.796	0.786	0.775	0.767
	10	0.772	0.766	0.758	0.753	0.747
	20	0.743	0.739	0.734	0.738	0.728
	30	0.737	0.732	0.728	---	---
2 (S)	0	1.348	1.348	1.348	1.348	1.348
	5	1.346	1.346	1.346	1.346	1.346
	10	1.346	1.346	1.346	1.346	1.346
	20	1.346	1.346	1.346	1.345	1.345
	30	1.345	---	---	---	---
3 (S)	0	3.738	3.738	3.737	3.736	3.734
	5	2.766	2.756	2.747	2.739	2.735
	10	2.691	2.685	2.680	2.675	2.672
	20	2.677	2.671	2.668	2.664	2.662
	30	2.673	---	---	---	---
4 (A)	0	3.529	3.524	3.511	3.493	3.469
	5	2.410	2.403	2.395	2.385	2.371
	10	2.404	2.400	2.392	2.379	2.368
	20	2.394	2.385	2.375	2.362	2.345
	30	2.366	2.356	2.343	---	---

* (S) symmetric mode; (A) antisymmetric mode

--- No results due to matrix ill-conditioning

4. FREQUENCY RESULTS AND COMPARISON STUDIES

Comparisons are made in Table 3 for the first six frequency parameters $wa^2\sqrt{\rho h/D}$ of deep (c/a = 0) and shallow (c/a = 0.75) notched square plates having a notch angle of 5° for a/h = 20. Frequency results obtained by using the present Mindlin plate theory are compared with those reported in a classical thin plate Ritz analysis [Ref. 5]. Here, it is seen that there is close agreement between the present Mindlin plate theory and classical thin plate theory, giving lower $wa^2\sqrt{\rho h/D}$ values for Mindlin notched plates due to the inherent shear deformation and rotary inertia.

Table 3 Comparison of frequency parameters $wa^2\sqrt{\rho h/D}$ for completely free square plates having a V-notch (a/b = 1, α = 355°, a/h = 20)

Mode no. (sym. class)*	c/a = 0		c/a = 0.75	
	CPT+	MPT++	CPT+	MPT++
1 (A)	2.095	2.074	3.304	3.206
2 (S)	3.785	3.720	4.841	4.446
3 (S)	5.542	5.461	6.003	5.865
4 (A)	5.982	5.820	8.367	8.000
5 (S)	8.161	7.886	8.587	8.175
6 (A)	9.449	9.126	15.14	14.23

* (S) symmetric mode; (A) antisymmetric mode
+Classical thin plate theory [Ref. 5]; ++Present Mindlin plate theory

5. CONCLUDING REMARKS

Highly accurate frequencies for completely free Mindlin square and rectangular plates with V-notches have been obtained using a Ritz procedure in conjunction with Mindlin plate theory. In this approximate procedure, the assumed displacements of the plate constitutes a hybrid set of complete algebraic-trigonometric polynomials along with Mindlin corner functions that account for singular bending moments and shear forces at the vertex of acute corner angles. The efficacy of such corner functions has been substantiated by a convergence study of non-dimensional frequencies.

The accuracy of the present frequency results have been examined through comparisons with thin plate solutions. Some fundamental understanding of the effect of highly localized stresses on the plate dynamics can be obtained through careful examination of the frequency data offered herein.

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