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An Analytical Solution of the Vertically One-dimensional Convection-Diffusion Equation for the Determination of Local Suspended Sediment Concentration
국지 부유퇴적물 농도의 결정을 위한 연직1차원 이류확산 방정식의 해석해

Kyung Tae Jung[1], Jae Youll Jin[1], Hyoun Woo Kang[1], Hong Yeon Cho[1], Mee Kyung Kim[1] and John Noye[2]
정경태[1] · 진재율[1] · 강현우[1] · 조홍연[1] · 김미경[1] · 존 노이[2]

1. INTRODUCTION

Convective-diffusion equations appear in various disciplines such as hydrology, chemical engineering and oceanography dealing with the transport problem of scalar quantities. Since it is nonlinear, numerical methods are generally used to obtain its solution. Very limited number of analytical solutions are available usually in cases when the convective velocity is constant or has a simple functional form (for some collection of the solutions, see Noye, 1987). There is however a continuing need to develop analytical solutions because of its practical importance. Analytical solutions of the convection-diffusion equation are valuable not only for the better understanding on the transport process but the verification of numerical schemes.

In this paper a vertically one-dimensional solution for the convection-diffusion equation will be presented, which are relevant to the study of the local time-varying structure of suspended sediment. The convection-diffusion equation was solved analytically by Dobbins (1944) to investigate the transient behaviors of the suspended sediment concentration from an initial distribution in the absence of the boundary flux. The solution was obtained using the classical technique of separation of variable. Okata and Banks (1961) presented an analytical solution of an initial value problem in a semi-infinite porous media.

A solution has been derived in this study in the presence of boundary flux using a transformation of dependent variable used by Okata and Banks(1961), and the eigenfunction expansion over the water column used by Heaps(1972) in oceanographic literature. The time-varying depth-variations of the suspended sediment concentration formed by shearing stresses of tidal current and wave are briefly included as application results.

2. BASIC EQUATION

We consider a horizontally infinite ocean of constant water depth and density. Assuming further that the eddy diffusivity is constant, we may write the basic equation governing the vertical distribution of suspended sediment in the following form.

1 한국해양연구원 연안항만공학본부 (Corresponding author: Kyung Tae Jung, Coastal and Harbor Res. Division, KORDI, Ansan PO Box 29, Seoul, ktjung@kordi.re.kr)

2 아델레이드대 응용수학과 (Department of Applied Math., Univ. of Adelaide, Adelaide, South Australia 5005)

$$\frac{\partial T}{\partial t} = - w\frac{\partial T}{\partial z} + \varepsilon\frac{\partial^2 T}{\partial z^2} \quad (1)$$

where t is time, z is the vertical coordinate ($z=0$ at the top of the water column and $z=-h$ at a reference level or sea bottom), $T(z,t)$ is the concentration of the suspended sediment, w is the settling velocity, which is negative, and ε is the vertical eddy diffusivity coefficient.

The boundary conditions at the top and bottom levels are taken as follows:

$$\begin{aligned} -w\,T(0,t) + \varepsilon\left(\frac{\partial T}{\partial z}\right)_0 &= 0, \\ -(w+v_d)T(-h,t) + \varepsilon\left(\frac{\partial T}{\partial z}\right)_{-h} &= -F_b. \end{aligned} \quad (2)$$

where F_b is the net flux of the suspended sediment at the sea surface, constant or time-varying, and v_d is a positive constant controlling the downward flux at the bottom. The subscripts 0 and $-h$ denote the level the partial derivative is evaluated. Note that the net flux at the sea surface is zero.

The system described by equation (1) and the boundary conditions (2) is, for convenience, modified to a equivalent system given below using delta function. In detail,

$$\frac{\partial T}{\partial t} = - w\frac{\partial T}{\partial z} + \varepsilon\frac{\partial^2 T}{\partial z^2} - F_b\,\delta(z+h) \quad (3)$$

with

$$\begin{aligned} -w\,T(0,t) + \varepsilon\left(\frac{\partial T}{\partial z}\right)_0 &= 0, \\ -(w+v_d)\,T(-h,t) + \varepsilon\left(\frac{\partial T}{\partial z}\right)_{-h} &= 0. \end{aligned} \quad (4)$$

Following Okata and Banks(1961) and also Stakgold (1972), we modify the equation (3) to a Fickian type diffusion equation. Introducing a transform of dependent variable in the form

$$T(z,t) = C(z,t)\cdot\exp\left(\frac{w}{2\varepsilon}z - \frac{w^2}{4\varepsilon}t\right), \quad (5)$$

we then have

$$\frac{\partial C}{\partial t} = \varepsilon\frac{\partial^2 C}{\partial z^2} - F_{bT}(t)\,\delta(z+h) \quad (6)$$

where

$$F_{bT} = \exp\left(-\frac{w}{2\varepsilon}z + \frac{w^2}{4\varepsilon}t\right)F_b(t)\,. \quad (7)$$

Along with the transformation of dependent variable given in equation (5), the boundary conditions given in equation (4) reduce to

$$\begin{aligned} -\frac{w}{2}C(0) + \varepsilon\left(\frac{\partial C}{\partial z}\right)_0 &= 0, \\ -\left(\frac{w}{2}+v_d\right)C(-h) + \varepsilon\left(\frac{\partial C}{\partial z}\right)_{-h} &= 0. \end{aligned} \quad (8)$$

3. SOLUTIONS

The Galerkin-eigenfunction method used by Heaps(1972) for the vertical variation of horizontal currents is then applied to decompose the diffusion equation into a set of ordinary differential equations which are first-order in time. The final form of solution is obtained by summing up the modal contribution and representing in the original variable. In what follows we describe full details of the solution procedure taken in this study.

3.1 Galerkin solution in terms of eigen-function expansion

We seek a solution for the diffusion equation (6) in the form

$$C(z,t) = \sum_{r=1}^{m} \mathcal{C}_r(t)\,f_r(z), \quad (9)$$

where $\mathcal{C}_r(t)$, $r=1,\cdots,m$, are the time-varying unknown coefficients, $f_r(z)$, $r=1,\cdots,m$, are a set of depth-varying basis functions. Initially the basis functions are assumed to be arbitrary.

To determine the time-dependent unknowns, the Galerkin approximation is applied. Taking first an inner product with f_k in equation (6) and applying integration by part twice to the vertical diffusion term (2nd term on the right hand side) gives

$$\int_{-h}^{0} \frac{\partial C}{\partial t} f_k dz = -\int_{-h}^{0} F_{bT}\delta(z+h) f_k dz + \left[\varepsilon \frac{\partial C}{\partial z} f_k\right]_{-h}^{0} - \left[\varepsilon C \frac{df_k}{dz}\right]_{-h}^{0} + \varepsilon \int_{-h}^{0} C \frac{d^2 f_k}{dz^2} dz \quad (10)$$

where $k = 1, \cdots, m$,

Incorporating the boundary conditions (8) and substituting the expansion given in (9) lead to

$$\sum_{r=1}^{m} \frac{\partial \widehat{C}_r}{\partial t} \int_{-h}^{0} f_r f_k dz = F_{bT} f_k(-h) - \sum_{r=1}^{m} \widehat{C}_r f_r(0) \cdot B_s - \sum_{r=1}^{m} \widehat{C}_r f_r(-h) \cdot B_b + \varepsilon \sum_{r=1}^{m} \widehat{C}_r \int_{-h}^{0} f_r \frac{d^2 f_k}{dz^2} dz \quad (11)$$

where

$$B_s = -\frac{w}{2} f_k(0) + \varepsilon \left(\frac{df_k}{dz}\right)_0$$
$$B_b = \left(\frac{w}{2} + v_d\right) f_k(-h) - \varepsilon \left(\frac{df_k}{dz}\right)_{-h} \quad (12)$$

Choosing f_k as a set of solutions (eigenfunctions) deduced from the well-known Sturm-Liouville system

$$\varepsilon \frac{d^2 f}{dz^2} + \lambda f = 0 \quad (13)$$

subject to

$$-\frac{w}{2} f(0) + \varepsilon \left(\frac{df}{dz}\right)_0 = 0$$
$$-\left(\frac{w}{2} + v_d\right) f(-h) + \varepsilon \left(\frac{df}{dz}\right)_{-h} = 0 \quad (14)$$
$$f(0) = 1$$

then the 2nd and 3rd terms of the right-hand side of equation (11) are eliminated and we get

$$\frac{\partial \widehat{C}_k}{\partial t} = \Phi_k F_{bT} f_k(-h) - \lambda_k \widehat{C}_k \quad (15)$$

where λ_k is the real-values k^{th} eigenvalue and Φ_k is the inverse of the norm of the eigenfunctions given by

$$\Phi_k = 1 / \int_{-h}^{0} f_k^2 dz \quad (16)$$

and the well-known orthogonality condition of the eigenfunctions, that is,

$$\int_{-h}^{0} f_r f_k dz = 0 \quad \text{if } r \neq k, \quad (17)$$

is used.

Assuming that the suspended sediment concentration is initially zero throughout the water column, we may write the solution of equation (15) as

$$\widehat{C}_k = \exp(-\lambda_k t) \cdot \int_0^t \exp(\lambda_k \tau) \cdot \Phi_k F_{bT}(\tau) f_k(-h) d\tau \quad (18)$$

In case the initial field of the suspended sediment is non-zero, it is necessary to expand the initial field in terms of the basis functions. Then, we get

$$\widehat{C}_k = \exp(-\lambda_k t) \cdot [\widehat{C}_k(0) + \int_0^t \exp(\lambda_k \tau) \cdot \Phi_k F_{bT}(\tau) f_k(-h) d\tau] \quad (19)$$

where

$$\widehat{C}_k(0)=\Phi_k\int_{-h}^{0}T(z,0)\exp\left(-\frac{w}{2\varepsilon}z\right)f_k dz \quad (20)$$

Eigenfunctions and eigenvalues determined from equations (13) and (14) are given as

$$f_k = \cos\alpha_k z+\frac{w}{2\varepsilon\alpha_k}\sin\alpha_k z$$
$$\lambda_k = \varepsilon\alpha_k^2 \qquad (k=1,\cdots,m) \quad (21)$$

where α_k satisfies the following transcendental equation.

$$v_d\cos\alpha_k h = \left\{\left(\frac{w}{2}+v_d\right)\frac{w}{2\varepsilon\alpha_k}+\varepsilon\alpha_k\right\}\cdot\sin\alpha_k h \quad (22)$$

The values of α_k, $k=1,\cdots,m$, in the above equation can be evaluated iteratively. In the presence of the non-zero settling velocity, the first eigenvalue is non-zero.

3.2 Solutions in terms of original variables

Summing up now the contribution of all eigenfunctions and substituting equation (18) into equation (9) give

$$C(z,t)=\sum_{r=1}^{m}f_r(z)\cdot\int_0^t\exp[\lambda_r(\tau-t)]\,\Phi_k F_{bT}(\tau)f_r(-h)\,d\tau \quad (23)$$

Substituting equations (21) and (23) into equation (5) finally gives

$$T\ (z,t)=\exp\left(\frac{w}{2\varepsilon}z-\frac{w^2}{4\varepsilon}t\right)\sum_{r=1}^{m}\left(\cos\alpha_r z+\frac{w}{2\varepsilon\alpha_r}\sin\alpha_r z\right)\cdot\left\{\int_0^t\exp[\varepsilon\alpha_r^2(\tau-t)]\,\Phi_r\cdot F_r(\tau)\,d\tau\right\} \quad (24)$$

where

$$F_r(\tau) = \exp\left[\frac{w}{2\varepsilon}h+\frac{w^2}{4\varepsilon}\tau\right]\cdot F_b(\tau)f_r(-h) \quad (25)$$

Integration of equation (24) can be computed analytically in case the erosion rate is defined as a constant or in a simple functional form, otherwise, is computed numerically using a mid-ordinate method.

4. APPLICATIONS

Two applications are briefly described. One is the calculation of the time-varying local structure of the suspended sediment in the presence of the oscillatory tidal shearing stress, another is the calculation in the presence of shearing stresses by tidal current and wave action.

4.1 Local time-varying structure of the suspended sediment due to tidal shearing stress

The water depth is in this study assumed to be $10m$ and the number of expansion functions m is taken as 16. The settling velocity w and the depositional velocity v_d are set to $-0.0005\ m/s$ and $0.00025m/s$ (that is, half of the absolute value of the settling velocity), respectively. The vertical eddy diffusivity coefficient ε is taken as $0.003m^2/s$.

It is worth to examine the forms of eigenfunctions which has been determined from the Sturm Liouville system with the mixed type boundary conditions (Fig. 1).

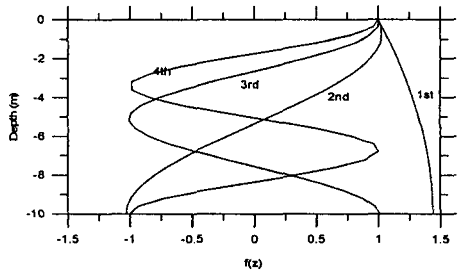

Fig. 1. Forms of the first four eigenfunctions computed from Sturm Liouville system of equation (14).

It is interesting to note from Fig. 1 that, unlike the case with homogeneous boundary conditions, the 1st eigenfunction increases downward. The r^{th} eigenfunction has exactly $r-1$ zero-crossings. The signs of $f_r(-h)$ alternate; the even numbered eigenfunctions have negative values, while the odd numbered eigenfunctions have positive values.

The erosion rate at the sea bottom is given as follows:

$$E(t) = \beta(1-\frac{\tau_b}{\tau_{cr}}) \quad \text{if } \tau_b \leq \tau_{cr}$$
$$= 0 \quad \text{if } \tau_b > \tau_{cr} \tag{26}$$

where β is a constant, τ_b is the bottom stress, and τ_{cr} is the critical bottom stress. The bottom stress is represented in quadratic from. Assuming that the velocity varies in a sinusoidal manner, we write

$$\tau_b = \rho k_b u_{\max}^2 \sin^2 \frac{2\pi}{T_p} t \tag{27}$$

where ρ is the water density, k_b is the bottom stress coefficient, $u_{\max}$ is the amplitude of the oscillatory flow, T_p is the period of the oscillatory flow. For experiments in this study we set $\tau_{cr} = 0.01\rho k_b$. That is, the threshold velocity for the erosion of bottom sediment is assumed to be 0.1 m/s. Other inputs values related with the specification of the erosion rate are: $\beta = 0.00005$; $\rho = 1025\ kg/m^3$; $k_b = 0.0025$; $u_{\max} = 0.8\,m/s$; $T_p = 12.0$ hours. Calculations have been conducted over five tidal cycles with a time step of $T_p/3600$ seconds. The suspended sediment concentration is initially assumed to be zero.

Fig. 2 shows the time variations in the suspended sediment concentration at the three depth levels ($h=-10m$, $-6m$ and $0m$). (Note that the method given in this study provides information exactly at the sea surface and sea bottom). It is evident that the concentration varies in an oscillatory manner. Roughly speaking, the concentrations reach a quasi-steady state after two tidal cycles. The amplitude of the oscillations is as expected the largest at the sea bottom, the smallest at the sea surface.

Close examination reveals that there are significant phase differences in the time variations of suspended sediment concentrations at different depths. It has been also noted that some negative concentrations appear at the very initial stage probably due to well-known Gibbs phenomenon of the spectral approach.

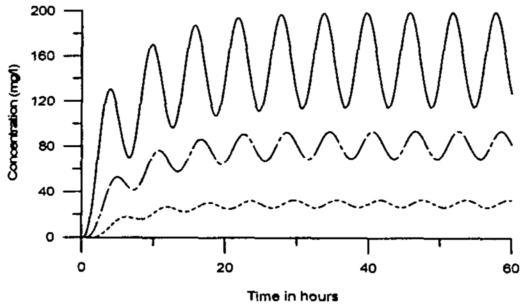

Fig. 2. Time variations in the suspended sediment concentration at $z=-h$(solid line), at $-0.6h$ (broken solid line) and $0.0h$ (dashed line).

Fig. 3 shows the vertical profiles of the suspended sediment concentration at times when the bottom concentration reaches at its maximum and minimum, respectively.

The maximum concentrations are found at the sea bottom in both cases but the values differ almost about two times. The minimum concentrations appear at the sea surface but their difference is very small. Although results are not shown here, calculations reveal that increasing the eddy diffusivity coefficient induces larger difference in the values at the sea surface. When the eddy diffusivity is decreased, the profile is sharpened, inducing a larger difference between the sea surface and sea bottom concentrations. That is, the eddy diffusivity determines the depth variation in the sediment concentration. On the contrary, the depositional velocity controls the overall concentration within the water column.

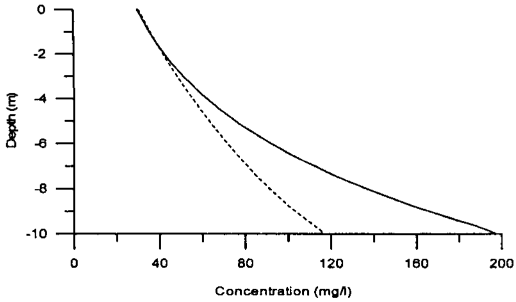

Fig. 3. The depth variations in the suspended sediment concentration when the bottom stress reaches its maximum (solid line) and minimum (dashed line).

4.2 Local time-varying structure of the suspended sediment in the presence of tidal current and wave action

Following Signell et al (1990) based upon the wave-current interaction model of Grant and Madsen (1976), we assume that

$$\tau_b = \tau_{cb} + \tau_{wb} \tag{28}$$

where τ_{tot} is the total bed shear stress, τ_{cb} is an instantaneous tidal current shear stress and τ_{wb} is the maximum wave bed stress given by

$$\tau_{wb} = \frac{1}{2} f_w \rho U_b^2 \tag{29}$$

where f_w is the wave friction factor, and U_b is the maximum near-bed wave orbital velocity.

The near-bed wave orbital velocity is given by

$$U_b = \frac{a_w \omega}{\sinh \varkappa h} \tag{30}$$

where a_w is the wave amplitude, ω is the wave frequency ($= 2\pi / T_w$), and $\varkappa$ is the wave number determined from the linear dispersion relation

$$\omega^2 = g\varkappa \tanh \varkappa h \tag{31}$$

where g is acceleration due to gravity.

The wave friction factor ω is computed from the formula of Johnsson (1967) and also Johnsson and Carlson (1976);

$$\frac{1}{4\sqrt{f_w}} + \log\left(\frac{1}{4\sqrt{f_w}}\right) = -0.08 + \log\left(\frac{U_b}{30\,\omega z_o}\right) \tag{32}$$

where z_o is roughness length. In this calculation we assume that the friction coefficient associated with tidal shearing stress remains unaffected by the presence of wave action. Equations (31) and (32) have been solved iteratively.

Three sets of calculations have been performed to examine the influence of wave amplitude, wave period and roughness height upon the buildup of the suspended sediment concentration field. It is again assumed that the sea bottom is the unlimited source of the suspended sediment. The tidal current shear has been computed as in 4.1.

Fig. 4 shows the time variations of the suspended sediment at the three depth levels, which are identical to previous results, computed using $a_w = 0.3m$ and $0.45m$. The wave period and the roughness height are follows:. $T_w = 7s$ and $z_o = 0.0001m$. As expected, increase in wave amplitudes gives rise to increase in the suspended sediment concentration at all depths. The concentration is approximately doubled. In this case contributions of the tidal shea and wave action to the sediment concentration are more or less same.

Fig. 5 compares the time variations of the suspended sediment concentrations computed using $T_w = 6s$ and $8s$ with $a_w = 0.3m$ and $z_o = 0.0001m$. It is evident that increase in wave period gives rise to increase in the suspended sediment concentration at all depths. We can note that, for a given value of wave amplitude, the increase in the wave period gives a higher value of orbital velocity at the sea bottom. Comparing with result of 4.1 (Fig. 2), we can see that wave action little changes the concentration when wave period is $6s$.

To examine the effects of roughness height, two values have been chosen: $z_o = 0.00005m$ and

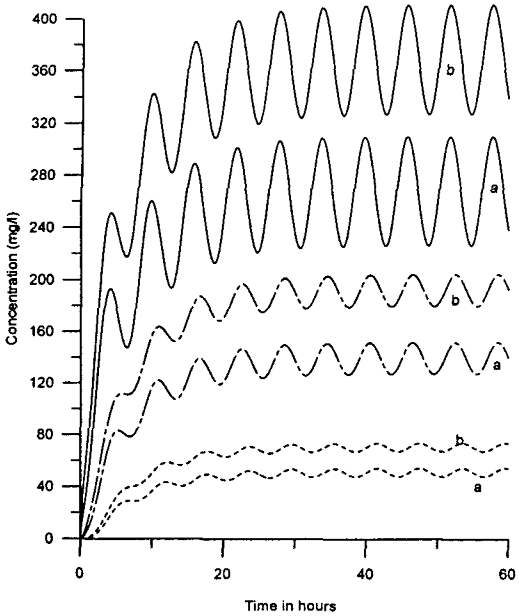

Fig. 4. Time variations in the suspended sediment concentration at $z=-h$(solid line), at $-0.6h$ (broken solid line) and $0.0h$ (dashed line) computed with tidal current and wave shears, using $T_w=7s$, and $z_o=0.0001m$, a) $a_w=0.3m$ and b) $0.45m$.

$0.0002m$, fixing the wave amplitude as $0.3m$. These values are roughly ten times smaller than ued by Davies and Lawrence (1994) in Irish experiments of tidal currents under wave-current interaction. The wave period and other parameters are same as before. Results are shown in Fig. 6. Increase in the concentration with a larger value of roughness height is expected because the wave friction factor is correspondingly increased (calculations with $z_o=0.00005m$ and $0.0002m$ give the friction factors $f_w=0.01565$ and 0.02719, respectively).

5. CONCLUSION AND DISCUSSION

In this study an analytical solution has been derived using the Galerkin-eigenfunction method. To accommodate the time-varying form of erosion rate, the solution has been presented in a time integral form. The solution of course deals with a more general situation than that considered by Dobbins

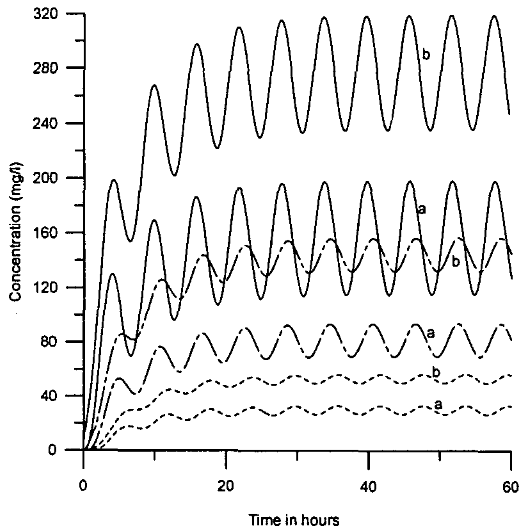

Fig. 5. Time variations in the suspended sediment concentration at $z=-h$(solid line), at $-0.6h$ (broken solid line) and $0.0h$ (dashed line) computed with tidal current and wave shears, using $a_w=0.3m$, and $z_o=0.0001m$. a) $T_w=6s$ and b) $8s$.

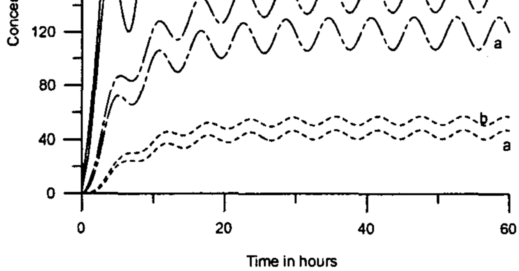

Fig. 6. Time variations in the suspended sediment concentration at $z=-h$(solid line), at $-0.6h$ (broken solid line) and $0.0h$ (dashed line) computed with tidal current and wave shears, using $a_w=0.3m$, and $T_w=7s$. a) $z_o=0.00005m$ and b) $0.0002m$.

(1944) and the solution methodology has more elegant features. The method presented in this study is applicable to the calculation of heat field in an idealized subduction region (see Jung et al, 2003).

The erosion rate has been defined in this study in terms of shearing stresses by tidal currents and/or wave action. There are two inherent limitations in the calculation. One is that the eddy diffusivity is assumed to remain constant, that is, the wave induced turbulence has been ignored, another is that the sea bottom is a unlimited source of the suspended sediment. Nevertheless, some sensitivity of the input parameters could be examined. It has been found that the suspended sediment sensitively varies according to wave amplitude, wave period and bottom roughness. Increase in the wave amplitude and period and the roughness height leads to the increase of the suspended sediment concentration, indicating that in a shallow coastal water the wave action can be much more important than tidal shearing effect.

A comment on the convergence of the solution might be of value. Since the eigenfunctions are global in nature, the convergence becomes slow as the water depth increases and the eddy diffusivity decreases.

Immediate concern is to extend the model to the situation the vertical eddy diffusivity varies in the vertical direction. For that, the eddy viscosity profile with a two-layer structure will be reported soon. Further expansion to two and three-dimensional models with a horizontal convection and diffusion will be also reported in the near future.

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