

A WEALTH-DEPENDENT INVESTMENT OPPORTUNITY SET: ITS EFFECT ON OPTIMAL CONSUMPTION AND PORTFOLIO DECISIONS *

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Abstract

We consider a consumption and investment problem where an investor's investment opportunity gets enlarged when she becomes rich enough, i.e., when her wealth touches a critical level. We derive optimal consumption and investment rules assuming that the investor has a time-separable von Neumann-Morgenstern utility function. An interesting feature of optimal rules is that the investor consumes less and takes more risk in risky assets if the investor expects that she will have a better investment opportunity when her wealth reaches a critical level.

Keywords : Consumption, investment, utility function, Brownian motion, optimal strategy, investment opportunity, critical wealth level.

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1 Introduction

It is important to understand investors' behavior in the analysis of financial market. Studies on consumption and investment problem of an economic agent usually assume that her available investment opportunity set is fixed throughout. However, it is not the case in practice that an investor has the same available investment opportunities all the time. As she gets richer, she is likely to find broader investment opportunities.

In this paper we study a consumption and investment problem assuming that an investor faces an enlarged investment opportunity set once her wealth level touches a critical level.

2 An investment problem

We consider a market in which there is a riskless asset and $m + n$ risky assets. We assume that the risk-free rate is a constant $r > 0$ and the price $p_0(t)$ of the riskless asset follows a deterministic process

$$dp_0(t) = p_0(t)r dt, \quad p_0(0) = p_0.$$

The price $p_j(t)$ of the j -th risky asset follows geometric Brownian motion

$$dp_j(t) = p_j(t) \left\{ \alpha_j dt + \sum_{k=1}^{m+n} \sigma_{jk} dw_k(t) \right\}, \quad p_j(0) = p_j, \quad j = 1, \dots, m+n,$$

where $\mathbf{w}(t) = (w_1(t), \dots, w_{m+n}(t))$ is a $(m+n)$ dimensional standard Brownian motion defined on the underlying probability space (Ω, \mathcal{F}, P) . We assume that the matrix $D = (\sigma_{ij})_{i,j=1}^{m+n}$ is nonsingular. Hence $\Sigma \equiv DD^T$ is positive definite. Let D_m denote the first m by $m+n$ submatrix of D , and $\Sigma_m = D_m D_m^T$, then Σ_m is the first m by m submatrix of Σ and is also positive definite. The controls are the nonnegative consumption rate $\mathbf{c} = (c_t)_{t=0}^\infty$ and the row vector process of fractions of wealth invested in the risky assets, $\pi = (\pi_t)_{t=0}^\infty$ which are adapted to $(\mathcal{F}_t)_{t=0}^\infty$, the augmentation under P of the natural filtration generated by the standard Brownian motion $(\mathbf{w}(t))_{t=0}^\infty$.

Let T_ξ be the first time that her wealth reaches ξ . There exists a critical wealth level z such that if time t is less than T_z then the investor is restricted to invest only in the riskless asset and the first m risky assets, but if $t \geq T_z$ then the investor can invest in all $m+n+1$ assets.

We let $\alpha = (\alpha_1, \dots, \alpha_{m+n})$ the row vector of returns of risky assets and $\mathbf{1}_{m+n} = (1, \dots, 1)$ the row vector of $m+n$ ones.

The investor faces a nonnegative wealth constraint

$$x_t \geq 0, \quad \text{for all } t \geq 0 \text{ a.s.}, \quad (2.1)$$

where x_t denotes the investor's wealth at time t .

We define an admissible set $A(x)$ by the set of control processes satisfying the above conditions with $x_0 = x$. The investor wishes to choose $(\mathbf{c}, \boldsymbol{\pi}) \in A(x)$ to maximize the expected total reward

$$V_{(\mathbf{c}, \boldsymbol{\pi})}(x) \equiv E_x \int_0^\infty \exp(-\beta t) U(c_t) dt$$

for $0 < x \equiv x_0 < z$, where E_x denotes the expectation operator conditioned on $x_0 = x$, the function U , called a utility function, is real-valued on $(0, \infty)$ and $\beta > 0$ is a discount rate. We assume that U is strictly increasing, strictly concave and three times continuously differentiable. We also assume that $\lim_{c \rightarrow \infty} U'(c) = 0$. For later use, we let $I(\cdot)$ be the inverse function of $U'(\cdot)$.

We let

$$V^*(x) \equiv \sup \{V_{(\mathbf{c}, \boldsymbol{\pi})}(x) : (\mathbf{c}, \boldsymbol{\pi}) \in A(x)\} \quad (2.2)$$

be the optimal expected reward or the optimal value at wealth x . Put

$$\kappa_1 \equiv \frac{1}{2}(\bar{\boldsymbol{\alpha}} - r\mathbf{1}_m)\Sigma_m^{-1}(\bar{\boldsymbol{\alpha}} - r\mathbf{1}_m)^T,$$

where $\mathbf{1}_m$ is the row vector of m ones and $\bar{\boldsymbol{\alpha}}$ denotes the m dimensional row vector consisting of the first m components of $\boldsymbol{\alpha}$. If we assume that $\kappa_1 > 0$, then the quadratic equation of λ

$$\kappa_1 \lambda^2 - (r - \beta - \kappa_1)\lambda - r = 0 \quad (2.3)$$

has two distinct solutions $\lambda_- < -1$ and $\lambda_+ > 0$. When the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, the optimal value at x , say $V_m(x)$, is finite and attainable by a strategy for all $x > 0$, as is shown in Karatzas, Lehoczky, Sethi, and Shreve [2], if it is assumed that

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty \quad (2.4)$$

for all $c > 0$. Similarly, put

$$\kappa_2 \equiv \frac{1}{2}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})^T.$$

If $\kappa_1 > 0$, then $\kappa_2 > 0$ and the quadratic equation of η

$$\kappa_2 \eta^2 - (r - \beta - \kappa_2)\eta - r = 0 \quad (2.5)$$

has two distinct solutions $\eta_- < -1$ and $\eta_+ > 0$. When the investment opportunity set consists constantly of all the $m + n + 1$ assets, the optimal value at x , say $V_{m+n}(x)$, is finite and attainable by a strategy for all $x > 0$ if it is assumed that

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} < \infty \quad (2.6)$$

for all $c > 0$. Thus, we assume the above conditions.

It is obvious that $V_{m+n}(z) \geq V_m(z)$. However, if $V_{m+n}(z) = V_m(z)$, the problem is trivial. Therefore we consider only the case

$$V_{m+n}(z) > V_m(z). \quad (2.7)$$

3 Optimal policies and their properties

For optimal policies and the value function, we consider the case where $U'(0) = \infty$. The case where $U'(0) < \infty$ is solved similarly: For $\hat{B} \geq 0$, define

$$X(c; \hat{B}) = \hat{B}(U'(c))^{\lambda_-} + X_0(c) \quad (3.1)$$

for $c > 0$, where

$$\begin{aligned} X_0(c) = & \frac{c}{r} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)} \left\{ \frac{(U'(c))^{\lambda_+}}{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \right. \\ & \left. + \frac{(U'(c))^{\lambda_-}}{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \right\}. \end{aligned} \quad (3.2)$$

Then, $X'(c; \hat{B}) > 0$ for all $c > 0$ and $X(\cdot; \hat{B})$ maps $[0, \infty)$ onto itself if we let $X(0) \equiv \lim_{c \downarrow 0} X(c)$. Let $C(\cdot; \hat{B})$ be the inverse function of $X(\cdot; \hat{B})$ and let $C_0 \equiv C(\cdot; 0)$, that is, C_0 is the inverse function of X_0 . For $\hat{A} \geq 0$, we also define

$$J(c; \hat{A}) = \hat{A}(U'(c))^{\rho_-} + J_0(c) \quad (3.3)$$

for $c > 0$, where

$$\begin{aligned} J_0(c) = & \frac{U(c)}{\beta} - \frac{1}{\kappa_1(\rho_+ - \rho_-)} \left\{ \frac{(U'(c))^{\rho_+}}{\rho_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \right. \\ & \left. + \frac{(U'(c))^{\rho_-}}{\rho_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \right\}, \end{aligned} \quad (3.4)$$

where $\rho_+ = 1 + \lambda_+$ and $\rho_- = 1 + \lambda_-$. If the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, then as is shown in Karatzas, Lehoczky, Sethi and Shreve [2], the optimal value at x becomes

$$V_m(x) = J_0(C_0(x)), \quad (3.5)$$

for $x > 0$ and an optimal strategy is given by

$$c_t = C_0(x_t), \quad \tilde{\pi}_t = \frac{V'_m(x_t)}{-x_t V''_m(x_t)} (\tilde{\alpha} - r \mathbf{1}_m) \Sigma_m^{-1}, \quad (3.6)$$

for $t \geq 0$ where $\tilde{\pi}_t$ denote the vector of fractions of wealth invested in the first m risky assets at time t . Define

$$\begin{aligned} X_{m+n}(c) = & \frac{c}{r} - \frac{1}{\kappa_2(\eta_+ - \eta_-)} \left\{ \frac{(U'(c))^{\eta_+}}{\eta_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\eta_+}} \right. \\ & \left. + \frac{(U'(c))^{\eta_-}}{\eta_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} \right\}, \end{aligned} \quad (3.7)$$

for $c > 0$. Then, $X'_{m+n}(c) > 0$ for all $c > 0$ and $X_{m+n}(\cdot)$ maps $[0, \infty)$ onto itself if we let $X_{m+n}(0) \equiv \lim_{c \downarrow 0} X_{m+n}(c)$. Let $C_{m+n}(\cdot)$ be the inverse function of

$X_{m+n}(\cdot)$. We also define

$$J_{m+n}(c) = \frac{U(c)}{\beta} - \frac{1}{\kappa_2(\rho_+ - \rho_-)} \left\{ \frac{(U'(c))^{\nu_+}}{\nu_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\eta_+}} \right. \\ \left. + \frac{(U'(c))^{\nu_-}}{\nu_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} \right\}, \quad (3.8)$$

where $\nu_+ = \eta_+ + 1$ and $\nu_- = \eta_- + 1$. Similarly to above, if the investment opportunity set consists constantly of all the $m + n + 1$ assets, then the optimal value at x becomes

$$V_{m+n}(x) = J_{m+n}(C_{m+n}(x)), \quad (3.9)$$

for $x > 0$ and an optimal strategy is given by

$$c_t = C_{m+n}(x_t), \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)} (\alpha - r \mathbf{1}_{m+n}) \Sigma^{-1}, \quad (3.10)$$

for $t \geq 0$. Let S be the $(m + n)$ dimensional row vector the first m components of which are equal to those of $(\bar{\alpha} - r \mathbf{1}_m) \Sigma_m^{-1}$ and the rest are equal to zero. We have the following theorem.

Theorem 3.1. *Suppose that $U'(0) = \infty$. Then, the value function is given by*

$$V^*(x) = J(C(x; \hat{B}); \frac{\lambda_-}{\rho_-} \hat{B}) \quad (3.11)$$

for $0 < x \equiv x_0 < z$, and an optimal policy is given by

$$c_t = C(x_t; \hat{B}), \quad \pi_t = \frac{V^{*'}(x_t)}{-x_t V^{*''}(x_t)} S \quad (3.12)$$

for $0 \leq t < T_z$, for a $\hat{B} > 0$, and

$$c_t = C_{m+n}(x_t), \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)} (\alpha - r \mathbf{1}_{m+n}) \Sigma^{-1} \quad (3.13)$$

for $t \geq T_z$, where $V_{m+n}(\cdot)$ is given by (3.9).

Proposition 3.1. *When $U'(0) = \infty$,*

$$C(x; \hat{B}) < C_0(x) \quad (3.14)$$

for $0 < x < z$, where $C(x; \hat{B})$ is given in Theorem 3.1 and $C_0(x)$ in (3.6).

Proposition 3.2. $\frac{V^{*'}(x)}{-V^{*''}(x)} > \frac{V'_m(x)}{-V''_m(x)}$ for $0 < x < z$.

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