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최대 로트 그룹핑 문제의 복잡성

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On the hardness of maximum lot grouping problem

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Abstract

We consider the problem of grouping orders into lots. The problem is modelled by a graph $G=(V,E)$, where each node $v \in V$ denotes order specification and its weight $w(v)$ the orders on hand for the specification. We can construct a lot simply from orders of single specification. For a set of nodes (specifications) $\theta \subseteq V$, if the distance of any two nodes in θ is at most d, it is also possible to make a lot using orders on the nodes. The objective is to maximize the number of lots with size exactly λ. In this paper, we prove that our problem is NP-Complete when $d=2, \lambda=3$ and each weight is 0 or 1. Moreover, it is also shown to be NP-Complete when $d=1, \lambda=3$ and each weight is 1, 2 or 3.

1. Introduction

In modern manufacturing, it is the usual case that orders in large variations of specifications come with small quantities. In order to handle this situation, we often make production facilities flexible to process orders with different specifications at the same time. Some orders with similar specifications can be grouped to a lot for production. For instance, in steel industry, orders of charges are specified by the ranges of widths and thicknesses. When the specifications on widths and thicknesses are not different significantly in dimension, orders of the specifications can be consolidated to be processed simultaneously.

We formally model our problem by a graph. A graph is a pair $G=(V,E)$, where V is a finite set of *nodes* and E has as elements sets of two nodes in V called *edges*. Each node v

corresponds to an order specification and its weight $w(v)$ denotes the order quantity on hand for the specification.

The *distance* of two nodes u and v is one if $\{u, v\} \in E$. In general, the distance between u and v is the distance of the shortest path from u to v. This can be found easily using the Floyd-Warshall algorithm (Floyd, 1962, Papadimitriou & Steiglitz, 1988, Warshall, 1962). Then, orders with specifications u and v can be grouped if the distance is no greater than the allowable *distance limit* of d. A *lot type* is any subset θ of V. If the cardinality of θ is one, it is called *homogeneous* otherwise called *heterogeneous*. A heterogeneous lot type θ is *feasible* if the distance of any two nodes in θ is at most d. Then, given a *lot size* λ (a positive integer), a *lot* with respect to the type θ is the order quantities corresponding to the nodes (specifications) in θ. A lot is *feasible* if its type θ is feasible and the total sum of the order quantities from the nodes in θ is exactly λ. The objective function is to maximize the number of lots, whether homogeneous or heterogeneous, with size λ. We call our problem *maximum lot grouping problem* or maximum grouping problem in short. Notice that if $\lambda = 2, d = 1$, and every $w(v) = 1,\ v \in V$, our problem is the same as maximum matching problem, for which optimal algorithms are provided (Micali & Vazirani, 1980, Papadimitriou & Steiglitz, 1988).

In this paper, we consider the hardness of the maximum grouping problem. The most key factor determining the problem's intractability would be distance limit more than anything else of other parameters of lot size and weight. Firstly, we will show that the problem with $d = 2$ is NP-Complete even when the graph G is bipartite, $\lambda = 3$ and every $w(v) = 0$ or $w(v) = 1$ for $v \in V$. Next, we deal with the case of $d = 1$. The problem with $d = 1$ is proved to be NP-Complete in general. And it is still hard even though the graph G is bipartite, $\lambda = 3$, and every $w(v) = 1, 2$ or 3 for $v \in V$. In the next section, a mathematical formulation is given to describe the problem explicitly and in section 3, the hardness of the maximum grouping problem is proved. Finally, conclusion follows in section 4.

2. Problem definition

For a feasible lot type $\theta \subseteq V$, its lot is denoted by x_θ where $x_\theta(v)$ is the quantity from the node $v \in \theta$. Note that $\sum_{v \in \theta} x_{\theta(v)} = \lambda$. The following lemma presents a useful result which allows us to make homogeneous lots as many as possible from heterogeneous lots.

<u>**Lemma 1**</u>

Given two heterogeneous feasible lots, $\bar{x}_\theta$ and $\hat{x}_\theta$, with respect to the same type θ, $\theta \geq 2$, we can make two feasible contracted lots $x_{\theta'}$ and $x_{\theta''}$, $x_{\theta'}, x_{\theta''} \subseteq \theta$, where $\theta' \subseteq \theta$ or $\theta'' \subseteq \theta$.

Proof.

Let $\theta = \{v_1, \ldots, v_k\}$, $k \geq 2$. Let v_i be the first node such that

$$\sum_{j=1}^{i-1}(\bar{x}_\theta(v_j) + \hat{x}_\theta(v_j)) < \lambda \text{ and}$$

$$\sum_{j=1}^{i}(\bar{x}_\theta(v_j) + \hat{x}_\theta(v_j)) \geq \lambda.$$

Then, we make a lot $x_{\theta'}$ where $\theta' = \{v_1, \ldots, v_i\}$ and

$$x_{\theta'}(v_j) = \bar{x}_\theta(v_j) + \hat{x}_\theta(v_j) \text{ for } j = 1, \ldots, i-1,$$

$$x_{\theta'}(v_i) = \lambda - \sum_{j=1}^{i-1}(\bar{x}_\theta(v_j) + \hat{x}_\theta(v_j)).$$

Next, we will construct another lot $x_{\theta''}$. First, consider the case of $\bar{x}_\theta(v_i) + \hat{x}_\theta(v_i) - x_{\theta'}(v_i) > 0$. In this case, we let $\theta'' = \{v_i, \ldots, v_k\}$ and construct $x_{\theta'}$ with $x_{\theta''}(v_i) = \bar{x}_\theta(v_i) + \hat{x}_\theta(v_i) - x_{\theta'}(v_i)$ and $x_{\theta''}(v_j) = \bar{x}_\theta(v_j) + \hat{x}_\theta(v_j)$ for $j = i+, \ldots, k$. Next, consider the case of $\bar{x}_\theta(v_i) + \hat{x}_\theta(v_i) - x_{\theta'}(v_i) = 0$. Similarly, we let $\theta'' = \{v_{i+1}, \ldots, v_k\}$ and $x_{\theta''}(v_j) = \bar{x}_\theta(v_j) + \hat{x}_\theta(v_j)$ for $j = i+1, \ldots, k$. Then, we can see that $\theta' \subset \theta$ or $\theta'' \subset \theta$ with $\sum_{v \in \theta'} x_{\theta'}(v) = \sum_{v \in \theta''} x_{\theta''}(v) = \lambda$. Hence, $x_{\theta'}$ and $x_{\theta''}$, are feasible contracted lots with respect to type θ.

In most case of the real manufacturing, it is suggested that we make homogeneous lots as many as possible rather than heterogeneous ones. Applying Lemma 1 continuously to all heterogeneous lots, we can finally get the desired solution that no more than one lot is constructed for each heterogeneous type (though several lots are possibly constructed for each homogeneous type). In addition, from the lemma, we can assure that there always exists an optimal solution such that no more than one heterogeneous lot was constructed for each heterogeneous type. Hence, it is enough to use a binary integer variable for each heterogeneous type to describe the amount of lot constructions. Let Θ be a set of feasible heterogeneous types. Then, for a type $\theta \in \Theta$, we use the binary variable y_θ to identify whether a lot has been constructed or not. Note that for a heterogeneous type θ, we have $y_\theta = 1$ iff $\sum_{v \in \theta} x_\theta(v) = \lambda$. The number of homogeneous lots for $\theta = \{v\}$ is just denoted by y_v. Then, we can formulate our problem as the following integer programming problem:

Maximize $\sum_{v \in V} y_v + \sum_{\theta \in \Theta} y_\theta$

Subject to

$$\lambda y_\theta - \sum_{v \in \theta} x_\theta(v) = 0 \qquad \theta \in \Theta$$

$$\lambda y_v - \sum_{\theta : v \in \theta} x_\theta(v) \leq w(v) \qquad v \in V$$

$$x_\theta(v) \geq 0,\ x_\theta(v)\text{: integer for } v \in \theta,\ \theta \in \Theta$$

$$y_v \geq 0,\ y_v \text{ : integer for } v \in V$$

$$y_\theta = 0 \text{ or } 1 \text{ for } \theta \in \Theta$$

3. The hardness of maximum lot grouping problem

We will show that the maximum grouping problem is NP-Complete, which further means that polynomial optimal algorithms cannot exist unless P=NP. Even in the restricted cases that a lot size, a distance limit and weights have small values or the graph is bipartite, the problem will be proven to be NP-Complete. To this purpose, we investigate the computational complexity of decision versions of the maximum grouping problem.

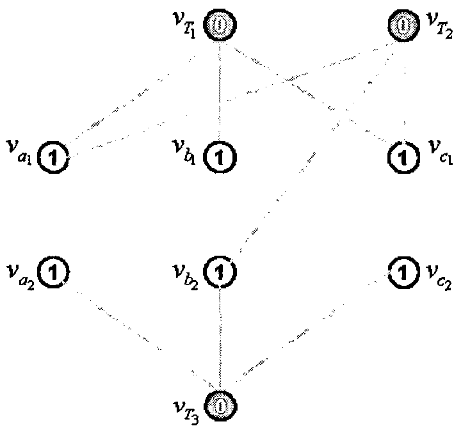
<u>**Theorem 2**</u> For the maximum lot grouping problem $G=(V,E)$ with $d=2, \lambda=3$ and each weight 0 or 1, the question of deciding if there exists number of $[\sum_{v \in V} w(v)/\lambda]$ feasible lots is NP-Complete, where $[x]$ denotes the smallest integer no greater than x.

We prove this result by showing that the known NP-Complete problem 3-dimensional matching can be transformed to the maximum grouping problem.

3-Dimensional Matching (3DM).
Instance : Disjoint sets $A=\{a_1,\dots,a_n\}$, $B=\{b_1,\dots,b_n\}$, $C=\{c_1,\dots,c_n\}$ and a family $F=\{T_1,\dots,T_m\}$ of triples with $|T_i \cap A| = |T_i \cap B| = |T_i \cap C| = 1$ for $i=1,\dots,m$.
Question : Does F contain a *matching*, that is, a subfamily F' for which $|F'|=n$ and $\cup_{T_i \in F'} T_i = A \cup B \cup C$?

$$A=\{a_1,a_2\} \quad T_1=\{a_1,b_1,c_1\}$$
$$B=\{b_1,b_2\} \quad T_2=\{a_1,b_2,c_1\}$$
$$C=\{c_1,c_2\} \quad T_3=\{a_2,b_2,c_2\}$$
$$F=\{T_1,T_2,T_3\}$$

(a)



(b)

Fig. 1. (a) A 3DM instance and (b) the corresponding graph.

For the sets A, B, C and F, we define corresponding node sets V_A, V_B, V_C and V_F as follows:
$V_A=\{v_{a_1},\dots,v_{a_n}\}$, $V_B=\{v_{b_1},\dots,v_{b_n}\}$,
$V_C=\{v_{c_1},\dots,v_{c_n}\}$, $V_F=\{v_{T_1},\dots,v_{T_m}\}$,

Proof of Theorem 2

Given an instance of the above 3DM problem, we construct an instance of the maximum lot grouping problem $G=(V,E)$ with lot size $\lambda = 3$, distance limit $d=2$. The node set V is given as follows:

$$V = V_A \cup V_B \cup V_C \cup V_F$$

Edges exist only when there is a corresponding triple in F:
edges are constructed between the nodes v_{T_i} and v_{a_j}, v_{b_k} or v_{c_l}, that is, three edges $\{v_{T_i}, v_{a_j}\}$, $\{v_{T_i}, v_{b_k}\}$ and $\{v_{T_i}, v_{c_l}\}$ are constructed if $T_i = \{a_j, b_k, c_l\}$ is in the family F. Now, we consider the weight of each node. We let $w(v_{a_j}) = 1$ $(w(v_{b_k}) = 1, w(v_{c_l}) = 1)$ for each $a_j \in A$ $(b_k \in B, c_l \in C)$, respectively. And for each $T_i \in F$, let $w(v_{T_i}) = 0$. In Fig. 1, the graph corresponding to an instance of 3DM is illustrated (the weight of each node is the number in the circle). Note that

$$n = \lfloor \sum_{v \in V} w(v)/\lambda \rfloor.$$

It is quite simple to show that there is number of n feasible lots if and only if there is a 3-dimensional matching. Suppose there is a matching F'. For each $T_i = \{a_j, b_k, c_l\} \in F'$, make a lot from the weights on the nodes v_{a_j}, v_{b_k} and v_{c_l}, i.e., a lot x_θ (with $y_\theta = 1$) for the type $\theta = \{v_{T_i}, v_{a_j}, v_{b_k}, v_{c_l}\}$ where $x_\theta(v_{T_i}) = 0$, $x_\theta(v_{a_j}) = x_\theta(v_{b_k}) = x_\theta(v_{c_l}) = 1$. Note that the distance between any two nodes of θ is at most two and the size of x_θ is three. Thus, x_θ is a feasible lot. Since there are n triples in the matching, we can make the n corresponding lots.

Conversely, suppose that there are n lots in G. As the lot size λ is three and each node has weight at most one, in the n lots there are no homogeneous ones. Note that, in V, for any two nodes u, v with distance at most two, there must exist a triple $T_i = \{a_j, b_k, c_l\} \in F$ such that $u, v \in \{v_{T_i}, v_{a_j}, v_{b_k}, v_{c_l}\}$. Then, recalling the distance
limit constraint $d=2$, we see that any feasible lot type θ must be a subset of nodes corresponding to a triple, say, T_i, that is, $\theta \subseteq \{v_{T_i}, v_{a_j}, v_{b_k}, v_{c_l}\}$. Let x_θ (with $y_\theta = 1$) be a feasible lot where $\theta \subseteq \{v_{T_i}, v_{a_j}, v_{b_k}, v_{c_l}\}$. Since the lot size is three and the weight of v_{T_i} is zero, θ is $\{v_{a_j}, v_{b_k}, v_{c_l}\}$ or $\{v_{T_i}, v_{a_j}, v_{b_k}, v_{c_l}\}$. Thus, for the lot x_θ, we have $x_\theta(v_{a_j}) = x_\theta(v_{b_k}) = x_\theta(v_{c_l}) = 1$ or $x_\theta(v_{a_j}) = x_\theta(v_{b_k}) = x_\theta(v_{c_l}) = 1$, $x_\theta(v_{T_i}) = 0$.

In either case, we see that for each lot x_θ, there exists exactly one corresponding triple T_i. Now, we choose n triples corresponding to the n lots. Note that each node v_{a_j} (v_{b_k}, v_{c_l}) has weight one. Thus, its weight or order quantity cannot be used in more than one lot, which means that the corresponding element a_j (b_k, c_l) does not belong to more than one triple of the chosen n triples. Therefore, we conclude the set of n triples is a matching.

Corollary 3

For the maximum lot grouping problem $G=(V,E)$ with $d=2, \lambda=3$ and each weight 0 or 1, the question of deciding if there exists number of $\lceil \sum_{v \in V} w(v)/\lambda \rceil$ feasible lots is NP-Complete even when G is bipartite.

Proof.

Let's consider again the graph $G=(V,E)$ corresponding to a 3DM in Theorem 2. Let $X=V_A \cup V_B \cup V_C$ and $Y=V_F$. Then, note that the node set V is partitioned into two disjoint sets X, Y and no edge exists between any two nodes in X and between any two nodes in Y, but edges exist between nodes in X and nodes in Y. Hence, G is a bipartite, proving the corollary.

Notice that when the distance limit d is one, one can find maximum number of lots using the maximum cardinality matching algorithm if $\lambda=2$, and every $w(v)=1$, $v \in V$ (Micali \& Vazirani, 1980, Papadimitriou \& Steiglitz, 1988). However, as we shall see in the following theorem, the maximum grouping problem is still hard in general even though the distance limit is one. The proof of the following theorem is almost similar to that of Theorem 3.7 in (Garey \& Johnson, 1979). We will transform our problem to 3DM as has been done in Theorem 2.

Theorem 4

For the maximum lot grouping problem $G=(V,E)$ with $d=1, \lambda=3$ and each weight 1, 2 or 3, the question of deciding if there exists number of $\lceil \sum_{v \in V} w(v)/\lambda \rceil$ feasible lots is NP-Complete.

Proof.

For the 3-dimensional matching problem, we construct an instance of the maximum grouping problem $G=(V,E)$ with lot size $\lambda=3$. The nodes and edges in the graph G will be specified from the the triples. For each triple, $T_i=\{a_j, b_k, c_l\}$, we construct a graph with edge set $E_i(|E_i|=6)$ as shown in Fig. 2, where the weight of each node is the number in the circle. Then, the node set V and edge set E of are given as follows:

$$V=(V_A \cup V_B \cup V_C) \cup_{i=1}^{m} \{u_{ij} : 1 \le j \le 4\},$$
$$E=\cup_{i=1}^{m} E_i.$$

Note that the total sum of weights of V is

$$\sum_{v \in V} w(v) = |V_A \cup V_B \cup V_C| + 9|F|$$
$$= 3(n+3m)$$

and $\lceil \sum_{v \in V} w(v)/\lambda \rceil = n+3m$.

We want to show that there is $n+3m$ feasible lots if and only if there is a 3-dimensional

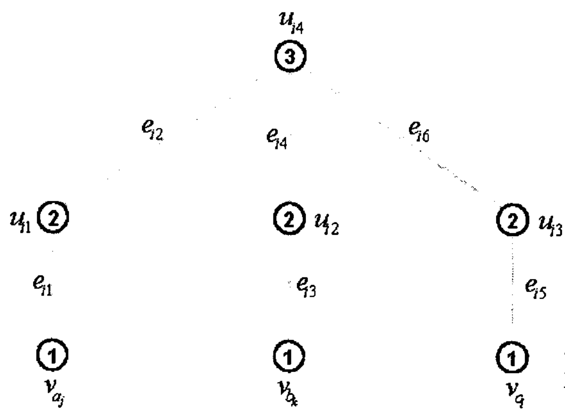


Fig. 2. The graph for the triple $T_i = \{a_j, b_k, c_l\}$.

matching. Suppose there is a matching F' from F for A, B and C. From this matching we can find number $n+3m$ lots from G, as described in the following: if $T_i = \{a_j, b_k, c_l\}$ is in the subfamily F', then the corresponding lots (three heterogeneous lots and one homogeneous lot) are given by

$y_{e_{i1}} = 1 \ (x_{e_{i1}}(v_{a_j}) = 1,\ x_{e_{i1}}(u_{i1}) = 2),$

$y_{e_{i3}} = 1 \ (x_{e_{i3}}(v_{b_k}) = 1,\ x_{e_{i3}}(u_{i2}) = 2),$

$y_{e_{i5}} = 1 \ (x_{e_{i5}}(v_{c_l}) = 1,\ x_{e_{i5}}(u_{i3}) = 2),$

$y_{u_{i4}} = 1$

otherwise if T_i is not in the subfamily F', then the corresponding lots (three heterogeneous lots) are given by

$y_{e_{i2}} = 1 \ (x_{e_{i2}}(u_{i1}) = 2,\ x_{e_{i2}}(u_{i4}) = 1),$

$y_{e_{i4}} = 1 \ (x_{e_{i4}}(u_{i2}) = 2,\ x_{e_{i4}}(u_{i4}) = 1),$

$y_{e_{i6}} = 1 \ (x_{e_{i6}}(u_{i3}) = 2,\ x_{e_{i6}}(u_{i4}) = 1).$

Conversely, suppose that there is a solution with $n+3m$ lots, that is, $\sum_{v \in V} y_v + \sum_{\theta \in \Theta} y_\theta = n+3m$. Then, the corresponding matching is given by choosing those $T_i \in F$ such that $y_{u_{i4}} = 1$.

Corollary 5

For the maximum lot grouping problem $G=(V,E)$ with $d=1, \lambda=3$ and each weight 1, 2 or 3, the question of deciding if there exists number of $[\sum_{v \in V} w(v)/\lambda]$ feasible lots is NP-Complete even when G is bipartite.

Proof.

Consider the maximum grouping problem $G=(V,E)$ corresponding to a 3DM in Theorem 4. Let X and Y be defined as follows:

$$X = V_A \cup V_B \cup V_C \cup_{T_i \in F} \{u_{i4}\},$$
$$Y = \cup_{T_i \in F} \{u_{i1}, u_{i2}, u_{23}\}.$$

Then, note that the node set V is partitioned into two disjoint sets X, Y and no edge exists between any two nodes in X and between any two nodes in Y, but edges exist between nodes in X and nodes in Y. Hence, G is a bipartite, proving the corollary.

4. Conclusion

In this paper, we introduced the maximum lot grouping problem. For the distance limit of two, the problem was shown to be NP-Complete even when the graph is bipartite, the lot size is three and each weight is 0 or 1. Next, we considered the case that the distance limit is one. Also in this case, the problem was proved to be still NP-Complete even when the graph is bipartite, the lot size is three and each weight is 1, 2 or 3.

It is open question whether an optimal algorithm exists for the case that $d = 1$ and the graph is grid. In general, we need to develop efficient approximation algorithms for maximum grouping problem.

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