한국경영과학회/대한산업공학회 2003 춘계공동학술대회
2003 년 5 월 16 일-17 일 한동대학교(포항)

수요, 주문 및 재고비용이 불확실한 상황에서의 EOQ 모형분석

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Analysis of EOQ Model Involving Estimate Errors

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Abstract

We consider the sensitivity of average inventory cost rate when true values of the parameters in the EOQ model are unknown over known ranges. In particular, in the case that the valid range on the true economic lot size are known, we provide a formula for estimating the lot size under minimax criterion. Moreover, to estimate the valid range, we apply the propagation of errors technique. Then, we present a scheme to find a (valid) lot size, based on the estimated range of the true lot size from the propagation of errors technique.

1. Introduction

In the basic Economic Order Quantity (EOQ) model, the optimal order quantity or lot size is determined by the three parameters of average demand rate, order (setup) cost, and inventory holding cost. When we know the exact values of the parameters, we can get the *true* (and optimal) lot size using the well-known EOQ formula. The values of the parameters are mainly measured in manufacturing or accounting departments. However, they often do not have the precise values but instead ranges for the estimated values of the parameters: the ranges in which they thought the true values might belong to. Hence, in practice we unavoidably have additional costs from the estimation errors, i.e., the difference between the average cost rate with precise values and the average cost rate with values of some errors.

In this situation, one methodology for making a decision on order quantity is to deploy the minimax criterion, likely to be used by risk-averse managers who desire to select alternatives that avoid the worst possible outcome. Many studies have been done for the sensitivity analysis of average cost rate to errors in parameter estimation (Groff and Muth, 1972, Lowe and Schwarz, 1983). In particular, Lowe and Schwarz provided an objective function to measure the effect of errors in parameter estimation: the ratio of the average cost rate with imprecise values to the average cost rate with true values, denoted by $R(Q)$. Then, the policy for decision making is to determine the lot size which minimizes the maximum of $R(Q)$.

However, it is questionable whether the lot size generated from the policy is within the *valid* range of the true size. Given a range to which the true optimal lot size belongs, it is trivial to check any suggested lot size, including the lot size from the policy, is valid or not. In case that no valid range is provided, we need to estimate the range of lot size using the estimated parameters of the EOQ model.

In this paper, we modify the policy to satisfy the validity constraint on lot sizes. To this end, a new formula for lot sizing has been derived to deal with the case when we are given a valid range of the true size. Moreover, in order to provide an estimate of the valid range, we deploy the propagation of errors technique. Illustrative examples will be presented to show the applicability of the technique. Finally, we present the scheme that finds a (valid) lot size, based on the estimated range of the true lot size from the propagation of errors technique.

2. Sensitivity of the EOQ Model

We are given ranges of estimations for the parameters, demand rate D, setup cost S, and inventory holding cost h as follows:

$$\begin{array}{ccccc} \underline{D} & \le & D & \le & \overline{D} \\ \underline{S} & \le & S & \le & \overline{S} \\ \underline{h} & \le & h & \le & \overline{h} \end{array} \qquad (1)$$

When one requires the amount of Q in each time of order, the average cost rate, $ACR(Q)$, is

$$ACR(Q) = (SD/Q) + (hQ/2).$$

The minimum of the average cost rate is attained at the lot size of $Q^* = \sqrt{2SD/h}$ with the cost rate

$$ACR(Q^*) = \sqrt{2SDh}. \qquad (2)$$

If one uses $\hat{Q}$ instead of Q due to estimation errors, additional costs incurs. To measure how much cost is increased, we use the ratio of the average cost ratio for $\hat{Q}$ to the average cost ratio for Q^*:

$$R(\hat{Q}) = \frac{ACR(\hat{Q})}{ACR(Q^*)}.$$

Let $\overline{K}$ be the set of triples (S,D,h) which satisfy the constraint (1). Though each parameter can have any value in its range, but all the triples in $\overline{K}$ might not be valid because of the interactions between the parameters. Let $\phi(S,D,h)$ be the joint probability distribution function and $\phi(\cdot)$ be defined only for the triples $(S,D,h) \in K$. Note that the triple $(S,D,h) \in \overline{K}$ is not valid if $(S,D,h) \notin K$.

When the given parameters are unknown, it is natural to choose the alternative that minimizes the worst-case outcome. This is called *minimax* criterion. In this criterion, we find the $\hat{Q}$ satisfying:

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in K} R(\hat{Q}) \qquad (3)$$

As it is not easy to get the set K in practice, the set $\overline{K}$ was used instead of K in the Lowe & Schwarz's model.

2.1 Lowe & Schwarz's Model

Lowe & Schwarz (1983) considered the problem

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in\bar{K}} R(\hat{Q}) \tag{4}$$

and showed that the lot size minimizing the maximum risk is $\hat{Q}^* = (4\overline{SD}\,\underline{SD})^{1/4}$. However, we note that the order quantity $\hat{Q}^*$ may not be valid, that is, no $(\hat{S},\hat{D},\hat{h}) \in K$ may exist with $\sqrt{2\hat{S}\hat{D}/\hat{h}} = \hat{Q}^*$. The following example shows some $\hat{Q}^*$ is not valid.

Example 1.

Let H^+ be the half space which includes all the points above or on the hyperplane crossing the three points $(\bar{S},\underline{D},\underline{h})$, $(\underline{S},\bar{D},\underline{h})$, $(\underline{S},\underline{D},\bar{h})$ (the point $(\bar{S},\bar{D},\bar{h})$ belongs to H^+). Then, the true set K is defined as $K = \bar{K} \cap H^+$. Let V be the volume of the hexahedron $\bar{K}$. The variables (S,D,h) are uniformly distributed with joint distribution function $\phi(\cdot)$:

$$\phi(S,D,h) = \begin{cases} \frac{1}{2V}, \text{ if } (S,D,h) \in K \\ 0, \text{ otherwise.} \end{cases}$$

When the ranges on the parameters, S, D, and h are $[60,90]$, $[100000,200000]$ and $[7,9]$, respectively, H^+ is the set $H^+ = \{(S,D,h): 200000S + 60D + 3000000h \ge 45000000\}$. In this case, the lot size satisfying (4) is $\hat{Q}^* = 1,618$. Then, we consider the minimum and maximum value of Q in the set K. Using an optimization tool MATLAB (The MathWorks, Inc. (2000)), we can get the minimum and maximum values, 1,630 and 2,267, respectively. We note here that the lot size $\hat{Q}^*$ is less than the minimum value. Hence, this lot size is not valid.

2.2 The Extended Model

Suppose that we are given a valid range for the true lot size,

$$\underline{Q} \le Q \le \bar{Q}, \tag{5}$$

as well as the ranges for the parameters of (1). Since the range (5) is valid, for each Q, $\underline{Q} \le Q \le \bar{Q}$, there exists at least one triple $(S,D,h) \in K$ such that $\sqrt{2SD/h} = Q$. We define another set of triples $\tilde{K}$ as

$$\tilde{K} = \left\{(S,D,h) : \underline{Q} \le \sqrt{2SD/h} \le \bar{Q},\ (S,D,h) \in \bar{K}\right\}.$$

In order to accommodate the validity information (5), the problem (4) is modified to

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in\tilde{K}} R(\hat{Q}). \tag{6}$$

Note that $ACR(\hat{Q}) = (SD/\hat{Q}) + (h\hat{Q}/2)$, which can be written as

$$ACR(\hat{Q}) = \sqrt{2SDh}\left[\frac{1}{\hat{Q}}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{SD}{h}}\right) + \hat{Q}\left(\frac{1}{2\sqrt{2}}\sqrt{\frac{h}{SD}}\right)\right]$$

.

Since $ACR(Q^*) = \sqrt{2SDh}$ by (2), from the definition of $R(\hat{Q})$, we have

$$R(\hat{Q})=\left[\frac{1}{\hat{Q}}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{SD}{h}}\right)+\hat{Q}\left(\frac{1}{2\sqrt{2}}\sqrt{\frac{h}{SD}}\right)\right]$$

Then, the problem (6) is equivalent to

$$\min_{\hat{Q}>0}\max_{(S,D,h)\in\tilde{R}}\left\{\left[\frac{1}{\hat{Q}}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{SD}{h}}\right)+\hat{Q}\left(\frac{1}{2\sqrt{2}}\sqrt{\frac{h}{SD}}\right)\right]\right\} \tag{7}$$

We let $y=SD/h$ with feasible region

$$\underline{y} \equiv \underline{Q}^2/2\le y\le\overline{Q}^2/2\equiv \overline{y}. \tag{8}$$

Then, we rewrite the problem (7) so that we consider only the variables $\hat{Q}$ and y:

$$\min_{\hat{Q}>0}\max_{\underline{y}\le y\le\overline{y}}\left\{2^{-1/2}y^{1/2}\hat{Q}^{-1}+2^{-3/2}y^{-1/2}\hat{Q}\right\}. \tag{9}$$

From the same arguments that Lowe and Schwarz (1983) used, we can easily show that the optimal solution to (9) is $\hat{Q}_R^*=(4\underline{y}\overline{y})^{1/4}$. Thus, from this together with (8), we have $\hat{Q}_R^*=\sqrt{\underline{Q}\,\overline{Q}}$. In the following proposition, the final result is summarized.

Proposition 1. The optimal solution to problem (6) is

$$\hat{Q}_R^*=\sqrt{\underline{Q}\,\overline{Q}}.$$

Then, the question remaining is how to get the estimated (valid) range of Q as in (5), which is the topic of the next section.

3. Error Analysis

Consider a function $y=f(x_1,\dots,x_n)$ where each variable x_j is defined for interval $(\underline{x}_j,\overline{x}_j)$ (in the EOQ model, $f(\cdot)$ is $f(S,D,h)=\sqrt{2SD/h}$). Then, we would like to get the valid range $(\underline{y},\overline{y})$ of the decision variable y. Since it is often the case that each mid value $x_j=(\underline{x}_j+\overline{x}_j)/2$ is thought of as the most likely one, we set the most likely value of y as $y=f(x_1,\dots,x_n)$. Let $\Delta y=\overline{y}-y=y-\underline{y}$, which is called the *composite error* from the estimation errors of the variables x_j. To estimate the composite error, two methods of the total differential and propagation of errors are often used. The composite error Δy in total differential method is defined as

$$\Delta y=\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\Delta x_j \tag{10}$$

In this equality, however, it is uncertain whether the effect of each individual error is to increase or decrease the combined error, which is a matter of randomness (Yoon, 1990). Hence, the range from the total differential method is so wide that it does not give us useful information.

In the propagation of errors technique, the error of y is not understood in terms of the approximate change to the disturbances of the variables x_j as in (10), but in terms of statistical deviation. From the statistical analyses in (Pugh and Winslow, 1996), for the standard deviation of y, we have

$$s_y^2 = \left(\frac{\partial f}{\partial x_1} s_1\right)^2 + \left(\frac{\partial f}{\partial x_2} s_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n} s_n\right)^2,$$

where σ_j is the standard deviation of variable x_j.

When we replace σ_y and σ_j's by Δy and Δx_j's, we have for the composite error

$$(\Delta y)^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_j} \Delta x_j \right)^2. \qquad (11)$$

Now, let's see how to apply the propagation of errors technique for the EOQ model, where the function $f(S,D,h) = \sqrt{2SD/h}$. In the EOQ model, for the corresponding composite error of (11), we have

$$\begin{aligned} \Delta Q^2 &= \left[\frac{\partial Q}{\partial S}\Delta S\right]^2 + \left[\frac{\partial Q}{\partial D}\Delta D\right]^2 + \left[\frac{\partial Q}{\partial h}\Delta h\right]^2 \\ &= \left[\sqrt{\frac{D}{2Sh}}\Delta S\right]^2 + \left[\sqrt{\frac{S}{2Dh}}\Delta D\right]^2 + \left[\sqrt{\frac{SD}{2h^3}}\Delta h\right]^2 \end{aligned} \qquad (12)$$

In Example 2, we take a look at how to compute the range of a lot size.

Example 2. – Deployment of Propagation of Errors Technique

A materials management department tries to find a lot size for a new part. It is best estimated that yearly demand is $D = (10{,}000 \pm 1{,}000)$, ordering cost $S = \$(100 \pm 10)$, and inventory carrying cost is $h = \$(10 \pm 1)$. Then, by (12), $\Delta Q = 39$. We estimate Q as $f(100, 10000, 10) = 447$. Thus, we obtain the range of Q as $(Q \pm \Delta Q) = (447 \pm 39)$ or $[408, 486]$.

The next example shows how well the propagation of errors technique works as opposed to actual statistics.

Example 3. – Comparison of Propagation of Errors Technique with Actual Statistics

The yearly demand is fixed with value of 10,000 and the other two estimates are given as follows:

- S: \$350, \$400, \$450 with equal probabilities and therefore a standard deviation of 123
- h: \$11, \$13, \$15 with equal probabilities and therefore a standard deviation of 2

Each of the nine joint probabilities from the two distributions has equal probability of 1/9. Thus, we have the following lot sizes with equal probabilities:

2763	2542	2366
	2717	2530
	2882	2683

The actual mean and standard deviation of these values are compared with those obtained by propagation errors:

Actual Statistic	Propagation of Errors
$\mu = 2{,}730$	$Q = 2{,}717$
$\sigma = 223$	$\Delta Q = 269$

As the comparison shows, estimations of propagation of errors technique is almost close to the actual statistics.

4. EOQ Decision with Propagation of Errors

Since the estimated range from the propagations of errors technique is somewhat quite accurate with the actual range, it is worthwhile to deploy the result of propagations of errors when making decisions. Thus, under minimax criterion, our scheme for lot sizing can be described as follows:

(1) Estimate the range on Q, $\underline{Q} \le Q \le \overline{Q}$, using the propagation of errors technique

(2) Calculate the lot size $\hat{Q}_R^*$ using the formula in Proposition 1.

To compare the result of our scheme ($\hat{Q}_R^*$) with that of Lowe & Schwarz's model ($\hat{Q}^*$), we provide the following example.

Example 4. – Comparison of $\hat{Q}_R^*$ and $\hat{Q}^*$

Consider Example 1 again with ten various instances of ranges of S, D and h. For each instance, we compared the lot size $\hat{Q}^*$ from Lowe & Schwarz's model with the lot size $\hat{Q}_R^*$ from our scheme based on the estimated valid range by propagation of errors. In Table 1, $(\underline{Q}, \overline{Q})$ is the true range found by an optimization tool MATLAB (The MathWorks, Inc. (2000)) and $(\underline{\hat{Q}}, \overline{\hat{Q}})$ is the estimated range by the propagation of errors technique. Note that that all the ten $\hat{Q}_R^*$'s are valid while the four of $\hat{Q}^*$ are not.

Table 1. Comparison of $\hat{Q}_R^*$ with $\hat{Q}^*$

No.	S		D		h		$\hat{Q}^*$	Q		$\hat{Q}$		$\hat{Q}_R^*$
	$\underline{S}$	$\overline{S}$	$\underline{D}$	$\overline{D}$	$\underline{h}$	$\overline{h}$		$\underline{Q}$	$\overline{Q}$	$\underline{\hat{Q}}$	$\overline{\hat{Q}}$	
1	200	300	1100	2000	17	33	175	156	266	134	218	171
2	10	90	100	10000	4	9	100	149	671	94	463	209
3	10	90	100	10000	1	9	141	149	1342	80	556	211
4	320	562	20000	40000	70	100	535	506	801	428	688	543
5	30	120	70000	100000	10	20	843	548	1549	595	1249	862
6	3000	5000	20000	40000	120	300	1075	889	1414	749	1389	1020
7	60	90	100000	200000	7	9	1618	1632	2267	1335	2019	1642
8	2	400	10	80000	1	2	189	400	8000	907	5641	2262
9	5000	8000	100000	200000	200	400	2515	2233	3162	1880	3219	2460
10	160	230	100000	200000	7	9	2615	2667	3625	2165	3243	2650

5. Conclusions

In the EOQ model with unknown values of demand rate, setup and carrying costs but instead with known ranges of them, we took sensitivity analysis into consideration. In extending the Lowe & Schwarz's model, under the minimax criteria, we derived new formula for generating lot size in case that we are given a valid range on the true lot size. To estimate the valid range, the propagation of errors technique has been used and its applicability was tested by examples. Finally, we suggested the scheme for lot sizing that first estimates the valid range using propagation of errors technique and then calculates the lot size from the new formula. Experiments showed that the scheme is more likely to generate valid lot sizes than the Lowe & Schwarz's model

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