# Gas Flow in a Rapidly Rotating Pipe with Azimuthal-Varying Thermal Wall Condition

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# 회전방향 온도변화를 갖는 매우 빠르게 회전하는 파이프 내의 기체유동

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**Key Words:** Compressible Rotating flow(압축성회전유동), Ekman number(에크만수), Energy transfer(에너지전달), Stewartson layer(스티워슨 경계층)

**Abstract**

An analysis on the steady-state has been made of flow of a compressible fluid rapidly-rotating in a pipe. The flow is induced by an small arbitrary azimuthally-varying thermal forcing added on the basic state of rigid body isothermal rotation. The system Ekman number is assumed to be very small value. Analytic solutions have been obtained for axisymmetric and non-axisymmetric types, in which the axisymmetric solution comes from the azimuthally-averaged wall boundary condition and the non-axisymmetric solution from fluctuating wall boundary condition.

## 1. Introduction

This paper is concerned with an analytical description of steady flow of a viscous, thermally-conducting compressible gas which is contained in a rapidly-rotating infinitely-long cylindrical pipe. The problem is characterized by the smallness of the Ekman number $E$ of the system, $E \ll 1$. In the basic state, the pipe rotates steadily about the longitudinal central axis at constant rotation rate $\Omega$, and the pipe and gas are in thermal equilibrium at constant temperature $T^*_{00}$. Here, the rotation rate is sufficiently high so that the compressibility effect, as represented by finite values of the Mach number $M$ of the fluid system, is significant. Also, the effective acceleration in the radial direction overwhelms the conventional earth's gravitational acceleration. Under these circumstances, the gas in the pipe is in rigid-body rotation, and the density increases exponentially in the radially-outward direction [ e.g., Sakurai & Matsuda (1974), Nakayama & Usui (1974), Bark & Bark (1976)]. The flow is now generated out of this basic-state of rigid-body rotation when a small temperature perturbation is imposed to the pipe wall. The practical relevance is this problem configuration is apparent in the design and operation of high-performance gas centrifuges [ e.g., Sakurai & Matsuda (1974)].

The findings of the present treatise are summarized for the case of a compressible fluid. Even when the external thermal forcing at the wall is non-axisymmetric, the fluid in the interior region maintains axisymmetric temperature distributions in the parametric

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range of $\sigma(\gamma-1)M^2 \gg O(E^{1/3})$. However, when $\sigma(\gamma-1)M^2 \lesssim O(E^{1/3})$, no z-independent solution can be obtained, and a fully z-dependent three dimensional flow results. In the case of $\sigma(\gamma-1)M^2 \gg O(E^{1/3})$, the interior fluid, by way of thermal diffusion process, approaches rigid-body rotation with its temperature equalized to the average temperature at the wall. On the other hand, the non-axisymmetric component of the thermal forcing at the wall induces a closed circulation in the $E^{1/3}$-thermal layer near the wall. During this process, heat is generated (removed) by compression (or expansion) work which is caused by radial flows. It is ascertained that the impact of non-axisymmetric thermal loading at the wall is absorbed within this boundary layer, and the temperature in the interior is substantially axisymmetric.

## 2. Mathematical Formulation

The basic state density field is

$$\rho_{00}(r) \equiv \exp\left[\frac{-\gamma M^2}{2}(r^2-1)\right] \quad (1)$$

The linearized governing Navier-Stokes equations, expressed in the cylindrical frame rotating at $\Omega^*$, can be written in nondimensional form as [e.g., Sakurai & Matsuda, 1976; Bark et. al. ; Park & Hyun, 2001]:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_{00}u) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho_{00}v) = 0, \quad (2)$$

$$-2\rho_{00}v - \gamma M^2 r\rho = -\frac{\partial p}{\partial r} + E\left[\left(\nabla^2 - \frac{1}{r^2}\right)u + \left(\frac{1}{3}+\beta\right)\frac{\partial}{\partial r}(\nabla\cdot\vec{u}) - \frac{2}{r^2}\frac{\partial v}{\partial\theta}\right] \quad (3)$$

$$2\rho_{00}u + \frac{1}{r}\frac{\partial p}{\partial\theta} = E\left[\left(\nabla^2 - \frac{1}{r^2}\right)v + \frac{1}{3r}\frac{\partial}{\partial\theta}\nabla\cdot\vec{u} + \frac{2}{r^2}\frac{\partial u}{\partial\theta}\right] \quad (4)$$

$$-\frac{\sigma(\gamma-1)}{\gamma}r\rho_{00}u = E\nabla^2 T, \quad (5)$$

$$p = \rho + \rho_{00}T. \quad (6)$$

It is convenient to deploy ^ to represent the azimuthal-averaged value of a dependent variable, i.e.,

$$\hat{\phi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(r,\theta,t)d\theta .$$

Furthermore, ~ denotes the departure of $\phi$ from $\hat{\phi}$,

$$\tilde{\phi} = \phi(r,\theta,t) - \hat{\phi}.$$

In fact, above notations ^ and ~ mean respectively axisymmetric and non-axisymmetric components of the variable.

It follows that arbitrary thermal forcing at the wall can be decomposed into axisymmetric and non-axisymmetric parts :

$$f(\theta)[\equiv T(r=1,\theta,t)] = \hat{f} + \sum_n \tilde{f}_n e^{in\theta},$$

where the index $n$ refers to the n-th complex Fourier coefficient

$$\tilde{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta .$$

Associate boundary conditions are expressed as :

(for the axisymmetric part)

$$\hat{u}(r,t=0) = \hat{v}(r,t=0) = \hat{T}(r,t=0) = 0 , \quad (7a)$$

and $\hat{u}(r=1,t)=0$ , $\hat{v}(r=1,t)=0$ ,

$$\hat{T}(r=1,t) = \hat{f} , \quad (7b)$$

(for the non-axisymmetric part)

$$\tilde{u}(r,\theta,t=0) = \tilde{v}(r,\theta,t=0) = \tilde{T}(r,\theta,t=0) = 0, \quad (7c)$$

and $\tilde{u}(r=1,\theta,t)=0$, $\tilde{v}(r=1,\theta,t)=0$,

$$\tilde{T}(r=1,\theta,t) = \sum_n \tilde{f}_n e^{in\theta}. \quad (7d)$$

## 3. Analysis

### 3.1 Axisymmetric part

The velocity and temperature fields in the steady state, subject to an axisymmetric thermal forcing, i.e., $T_w = \hat{f}$ , are now delineated [see Eq.(7a)]. The subscript $w$ denotes the pipe wall at $r=1.0$. . By imploying the averaging process to the continuity equation (2), the axisymmetric radial velocity is

$$\hat{u}_s(r) = 0. \quad (8)$$

Substituting Eq.(8) into azimuthally-averaged equations of Eqs.(4) & (5) yields

$$\hat{v}_s(r) = 0, \quad (9)$$

$$\hat{T}_s(r) = \hat{f} . \quad (10)$$

In the above, subscript s refers to the steady

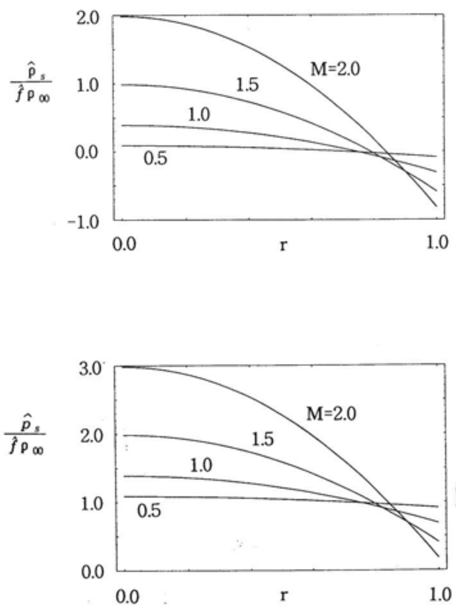


Fig.1 Axisymmetric fields of density [top frame] and pressure [bottom frame].

state. Obviously, the above results indicate that the steady-state flow is in isothermal rigid-body rotation.

The associated density and pressure fields can be found by undergoing algebraic manipulations. From Eq.(6), one has

$$\hat{p}_s(r) = \hat{\rho}_s(r) + \rho_{00}(r)\hat{f} \quad . \quad (11)$$

Bringing Eq.(11) into the axisymmetric part of Eq.(3) produces

$$\frac{d\hat{\rho}_s}{dr} - \gamma M^2 r \hat{\rho}_s = -\hat{f}\frac{d\rho_{00}}{dr} . . \quad (12)$$

The solution to Eq.(12) is found to be

$$\hat{\rho}_s(r) = \left(C_1 - \frac{\hat{f}}{2}\gamma M^2 r^2\right)\rho_{00}(r) \quad . \quad (13)$$

The integration constant $C_1$ is determined by using the global mass continuity, i.e., $\int_0^1 \hat{\rho}_s \cdot 2\pi r dr = 0$: (14)

$$C_1 = \frac{\hat{f}}{2}\left(\frac{\gamma M^2}{1 - e^{-\gamma M^2/2}} - 2\right) \quad (15)$$

The pressure $\hat{p}_s$ is obtained from Eq.(11) :

$$\hat{p}_s(r) = \frac{\hat{f}}{2}\gamma M^2\left(\frac{1}{1 - e^{-\gamma M^2/2}} - r^2\right)\rho_{00}(r) . (16)$$

The results are shown in Fig.1.

### 3.2 Non-axisymmetric part

For $E \ll 1$, to secure a meaningful asymptotic solution, the flow variables are expanded as

$$\tilde{\Phi} = \sum_{n=0}^{\infty} E^{n/m}\,\tilde{\Phi}_n(\zeta, \theta),$$

in which $\zeta = (1 - r)/E^{1/m}$ and $\tilde{\Phi}$ denotes $\tilde{u}, \tilde{v}, \tilde{\rho}, \tilde{p}$ or $\tilde{T}$. Upon substituting above expansions into non-axisymmetric parts of governing equations (2)-(6), one can find a choice of $m = 3$ and $\tilde{u}_0 = \tilde{p}_0 = 0$. Therefore, the problem has a proper expansion parameter of $E^{1/3}$ and the associated boundary layer coordinate as $\zeta = (1 - r)/E^{1/3}$. The meaningful leading-order dependent variables are scaled as

$\tilde{u}_s \sim O(E^{1/3})$, $\tilde{v}_s \sim O(1)$, $\tilde{T}_s \sim O(1)$,

$\tilde{\rho}_s \sim O(1)$, $\tilde{p}_s \sim O(E^{1/3})$.

In the above, tilde refers to a non-axisymmetric component and subscript s the steady solution. Above scalings are very similar to $E^{1/3}$-Stewartson layer [ Bark & Bark (1976)] and to $E^{1/3}$-thermal layer [Matsuda & Nakagawa (1983); Wood & Babarsky(1992)].

The leading-order governing equations for $\tilde{T}_s$ and $\tilde{v}_s$ are

$$\gamma M^2 \frac{\partial \tilde{T}_s}{\partial \theta} = \frac{\partial^3 \tilde{v}_s}{\partial \zeta^3}, \quad (17a)$$

$$-\frac{\sigma(\gamma-1)}{\gamma}\frac{\partial \tilde{v}_s}{\partial \theta} = \frac{\partial^3 \tilde{T}_s}{\partial \zeta^3}. \quad (17b)$$

By eliminating $\tilde{v}$ from Eqns.(17a) and (17b), the equation for temperature is obtained :

$$\frac{\partial^6 \tilde{T}_s}{\partial \zeta^6} + \sigma(\gamma-1) M^2 \frac{\partial^2 \tilde{T}_s}{\partial \theta^2} = 0.. \quad (18)$$

The appropriate boundary conditions for Eq.(18) are :

$$\text{at } \zeta = 0, \quad \tilde{T}_s = \sum_n \tilde{f}_n e^{in\theta}, \quad (19a)$$

$$\frac{\partial^2 \tilde{T}_s}{\partial \zeta^2} = 0, \quad (19b)$$

$$\frac{\partial^3 \tilde{T}_s}{\partial \zeta^3} = 0, \quad (19c)$$

and, as $\zeta \rightarrow \infty$, all the variables tend to zero. Eq.(19a) expresses the imposed thermal loading at the wall, and Eq.(19b) states the non-permeable condition at the wall, i.e., $\tilde{u}_s = 0$ leads to $\partial^2 \tilde{T}_s / \partial \zeta^2 = 0$, and Eq.(19c) states the no-slip condition at the wall , i.e., $\tilde{v}_s = 0$ leads to $\partial^3 \tilde{T}_s / \partial \zeta^3 = 0$.

In a manner similar to Sakurai and Matsuda (1974), the function $\tilde{T}$ is expanded as

$$\tilde{T}_s(\zeta, \theta) = \sum_n \tilde{T}_n(\zeta) e^{in\theta} \quad . \quad (20)$$

By substituting Eq.(20) into Eq.(18), the solution $T_n(\zeta)$, subject to eqs.(19a)-(19c), is acquired :

$$\tilde{T}_n(\zeta) = \frac{\tilde{f}_n}{2}\Big[ e^{-\gamma_n \zeta} + e^{-\frac{1}{2}\gamma_n \zeta} \frac{2}{\sqrt{3}} \cos\Big(\frac{\sqrt{3}}{2}\gamma_n \zeta - \frac{\pi}{6}\Big)\Big] \quad (21)$$

in which $\gamma_n = (\sigma(\gamma-1) M^2 n^2)^{1/6}$ .

The radial and azimuthal velocities are given :

$$\tilde{u}_s(\zeta, \theta) = -\frac{\gamma}{\sigma(\gamma-1)} \sum_n \tilde{U}_n(\zeta) e^{in\theta} \quad , \quad (22)$$

$$\tilde{v}_s(\zeta, \theta) = -\frac{\gamma}{\sigma(\gamma-1)} \sum_n \tilde{V}_n(\zeta) e^{i(n\theta-\pi/2)} \quad ,(23)$$

in which

$$\tilde{U}_n(\zeta) = \frac{\tilde{f}_n}{2}\gamma_n^2\Big[ e^{-\gamma_n \zeta} + \frac{2}{\sqrt{3}} e^{-\gamma_n \zeta/2} \sin\Big(\frac{\sqrt{3}}{2}\gamma_n \zeta - \frac{\pi}{3}\Big)\Big]$$

$$\tilde{V}_n(\zeta) = -\frac{\tilde{f}_n}{2n}\gamma_n^3\Big[ e^{-\gamma_n \zeta} + \frac{2}{\sqrt{3}} e^{-\gamma_n \zeta/2} \sin\Big(\frac{\sqrt{3}}{2}\gamma_n \zeta - \frac{2\pi}{3}\Big)\Big]$$

The primary role of the thermal layer is to adjust the interior flow smoothly to the conditions imposed at the wall. It is, therefore, sufficient to carry out only the leading-order analysis at this stage.

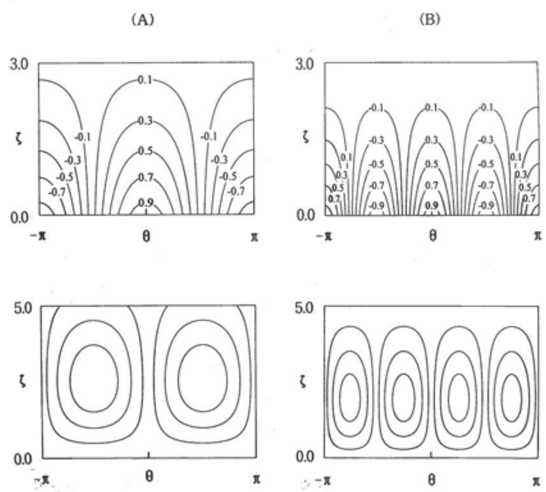


Fig.2 Plots of harmonic components for non-axisymmetric temperature fields (top frame) and stream function (bottom frame). (A), n=1; (B), n=2. $M = 1.0$, $\gamma = 1.4$ and $\sigma = 0.7$. $\Delta\Psi = 0.3$.

The above theoretical developments offer succint physical interpretations. As shown in Eq.(18), when $\sigma(\gamma-1) M^2 \gg O(E^{1/3})$, the effect of non-axisymmetric thermal forcing at the wall is absorbed in the thermal layer of thickness $O(E^{1/3})$ adjacent to the wall. Inside the thermal layer, the fluid near the wall where the thermal forcing is positive, i.e., in the azimuthal wall sector under heating, undergoes thermal expansion. This causes radially-inward flows [see eq.(22)]. In contrast, in the azimuthal wall sector under cooling, the fluid undergoes thermal compression, which generates radially-outward flows. In response to the non-axisymmetric thermal condition at the wall, i.e., when the azimuthal wall sectors under heating and cooling exist, the resulting radially-inward and neighboring radially-outward flows induce azimuthal flows along the wall within this $O(E^{1/3})$-layer [see eqs.(22) & (23)]. This is clear from the consideration of the continuity equation. [see Fig.2]

The overall picture inside this thermal layer is that, in the immediate vicinity of the wall, azimuthal flows are generated from a cool to a hot wall sector; in the far region away from the wall, azimuthal flows are in the opposite direction. These internal flows inside the layer create a closed circulation. In this process, the fluid experiences diffusive heating (cooling) from the hot (cold) sector of the wall. Simultaneously, the radially-inward (outward)-moving fluid at hot (cold) sector is cooled (heated) by way of work done by the basic pressure variation, i.e., $\frac{\sigma(\gamma-1)}{\gamma}\tilde{u}_s<0$ ($\frac{\sigma(\gamma-1)}{\gamma}\tilde{u}_s>0$), which is easily confirmed from the energy equation (5). This aspect is absent in the case of an incompressible fluid. For an incompressible fluid, the temperature field is governed by the simplistic diffusion equation. Therefore, a temperature disturbance at the wall is not confined to within the $O(E^{1/3})$ boundary layer; the entire flow domain is directly influenced by the thermal boundary condition at the wall. This qualitative difference between compressible and incompressible flows was emphasized earlier by Matsuda et al. (1976).

In the present discussion for a rapidly-rotating compressible fluid, under $\sigma(\gamma-1)M^2 \gg O(E^{1/3})$, the essential dynamical element is the generation of radial motions of $O(E^{1/3})$ due to the imposition of non-axisymmetric thermal forcing at the wall [Remember that $\tilde{u}_s \sim O(E^{1/3})$]. The radially-inward (outward) motion causes volume expansion (compression) of the fluid element owing to the basic-state background pressure distribution, which results in cooling (heating) of the fluid. It is noted that the direct conductive heating (cooling) from the hot (cold) sector of the wall is offset by the cooling (heating) due to the afore-said radial motions. Consequently, it is possible that the effect of non-axisymmetric thermal forcing at the wall is restricted to within the $E^{1/3}$-thermal layer, rather than propagating to the entire flow domain approaching $r\rightarrow 0$.

In the case of differential heating of the pipe with highly conducting wall, exemplary schematics for above descriptions are shown in the following Figs.3-5.

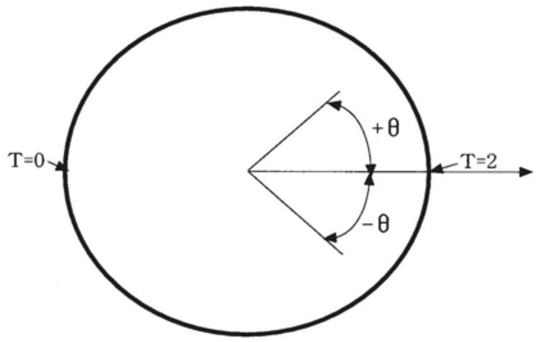


Fig.3 Problem definition of the differential heating problem.

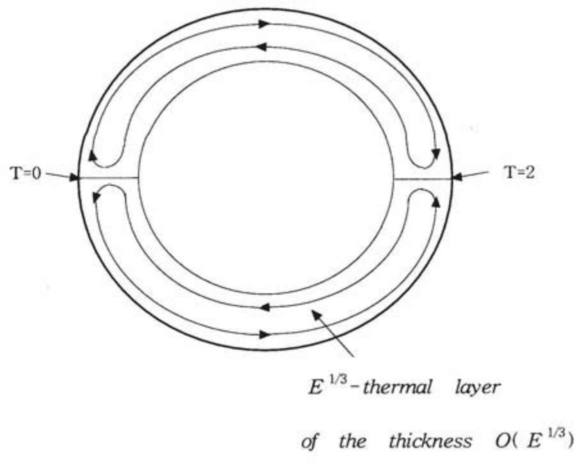


Fig.4 Schematics of flow geomery for the differential-heating problem.

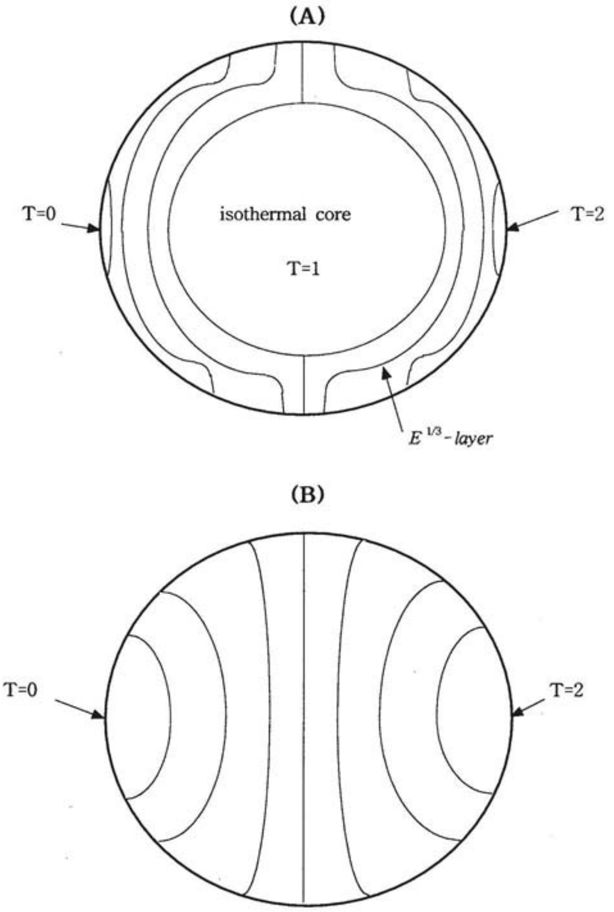


Fig.5 Schematics of isothermal lines in the case of (A), compressible fluid and (B), incompressible fluid.

## 4. Conclusions

The axisymmetric part of the thermal forcing at the wall gives rise to the interior temperature, which is equalized to the average value, $\hat{T}_w$, of the temperature distribution at the wall. The attendant flow in the interior is in rigid-body rotation, rotating together with the pipe. In comparison with the original basic-state, the density increases (decreases) in the interior region surrounding the axis for $\hat{T}_w>0$ ( $\hat{T}_w<0$). Near the wall, density decreases (increases) for $\hat{T}_w>0$ ( $\hat{T}_w<0$). The pressure increases (decreases) in the entire domain for $\hat{T}_w>0$ ( $\hat{T}_w<0$).

Due to a z-independent non-axisymmetric thermal forcing at the wall, in the parameter range $\sigma(\gamma-1)M^2 \lesssim O(E^{1/3})$, it is not possible to have a z-independent two-dimensional flow on the $(r,\theta)$ plane.

In the parameter range $\sigma(\gamma-1)M^2 \gg O(E^{1/3})$, forcing at the wall is absorbed in the $E^{1/3}$-thermal layer close to the wall. In the $E^{1/3}$-layer, regions of non-equal temperatures, deviating from the average value $\hat{T}_w$, are formed due to thermal diffusion from the wall. The accompanying azimuthally-varying density field gives rise to a closed circulation, which is in accord with thermal geostrophic wind relation.

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