Flow Near a Rotating Disk with Surface Roughness

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표면조도를 갖는 회전판 주위의 유동

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Key Words: Ekman layer(에크만 경계층), Rotating flow(회전유동), Rossby number(로스비수), Skin friction coefficient(표면마찰계수)

Abstract

It has been studied the flow near a rotating disk with surface topography. The system Ekman number is assumed very small, i.e., $E\left[\equiv \frac{\nu}{\Omega^{*} L^{*2}}\right] \ll 1$ in which L^{*} denotes a disk radius, ν kinematic viscosity of the fluid and Ω^{*} angular velocity of the basic state. Disk surface has a sinusoidal topographic variation along radial coordinate, i.e., $z = \delta \cos(2\pi \omega r)$, where δ and ω are, respectively, nondimensional amplitude and wave number of the disk surface. Analytic solutions, being useful over the parametric ranges of $\delta \sim O(E^{1/2})$ and $\omega \le O(E^{1/2})$, are secured in a series-function form of Fourier-Bessel type. An asymptotic behavior, when $E \to 0$, is clarified as : for a disk with surface roughness, in contrast to the case of a flat disk, the azimuthal velocity increases in magnitude, together with the thickening boundary layer. The radial velocity, however, decreases in magnitude as the amplitude of surface waviness increases. Consequently, the overall Ekman pumping at the edge of the boundary layer remains unchanged, maintaining the constant value equal to that of the flat disk.

1. Introduction

Flow of a viscous fluid near a disk of radius L^{*}, which rotates steadily about its axis, poses a classical problem. Specifically, consideration is given to the situation in which the disk rotates with uniform angular velocity $\Omega^{*} + \Delta\Omega^{*}$, while the fluid far away from the disk rotates with angular velocity Ω^{*}. The strength of the resulting flow is gauged by the Rossby number $\varepsilon \equiv \Delta\Omega^{*}/\Omega^{*}$. For the case of a flat disk, and, for the linearized flow $\varepsilon \ll 1$, the descriptions of the three-component velocity field of the Ekman layer solution are well-documented [see. e.g., Greenspan, 1968]. The underlying assumption is that the direct effect of viscosity on the global flow reaches at small distance from the disk, which is reflected in the smallness of the Ekman number $E\left[\equiv \frac{\nu}{\Omega^{*} L^{*2}}\right] \ll 1$, in which ν denotes the kinematic viscosity of the fluid.

It is emphasized here that the majority of previous fundamental studies on Ekman layer flow have been concerned with a flat disk. The disks with surface roughness have not

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been treated in sufficient details. In view of the prevalence of non-flat disks in industrial applications, the flow features close to the rotating disks with rough surfaces warrant on in-depth scrutiny. Knowledge of the flow and transport characteristics caused by the rotating non-flat disk is essential in the design and operation of heat exchangers and high-performance fluid machinery.

As a first step toward more complex systems, in the present paper, the surface roughness is modeled by an axisymmetric sinusoidal topography, as sketched in Fig.1. Efforts are made to acquire analytical descriptions of the flow under the assumption that the surface roughness is generally mild.

In chapter 2, the mathematical formulation is provided. Analyses are presented in chapter 3, which include the theoretical solutions for both a flat disk and a wavy disk. The asymptotic solution in the limit $E \to 0$ is considered in chapter 4. The concluding remarks are given in chapter 5.

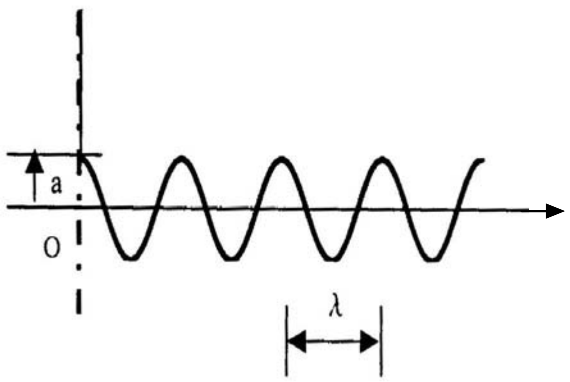

Fig.1 Configuration of the wavy disk

2. Mathematical Formulation

The problem is formulated on a cylindrical frame (r^*, θ, z^*), with the corresponding velocity components (u^*, v^*, w^*), which rotates uniformly about the z^*-axis with the rotation rate Ω^*. The dimensionless variables (unstarred) are introduced as follows :

$$(u, v, w) = \frac{1}{\varepsilon \Omega^* L^*}(u^*, v^*, w^*),$$

$$(r, z) = \frac{1}{L^*}(r^*, z^*),$$

$$p = \frac{p^*}{\varepsilon \rho^* \Omega^{*2} L^{*2}},$$

in which p and ρ denote, respectively, the pressure and density.

For $\varepsilon \ll 1$, the departure from the rigid-body rotation is small, and, as viewed from the rotating frame, the dependent variables are $O(\varepsilon)$. Summarizing the above considerations, the linearized nondimensional governing equations for Ekman boundary layer emerge :

$$\frac{1}{r}\frac{\partial}{\partial r}(r u) + E^{1/2}\frac{\partial w}{\partial z} = 0, \quad (1)$$

$$-2v = E\left(\nabla^2 - \frac{1}{r^2}\right)u, \quad (2)$$

$$2u = E\left(\nabla^2 - \frac{1}{r^2}\right)v, \quad (3)$$

$$-\frac{\partial p}{\partial z} + E^{1/2}\nabla^2 w = 0, \quad (4)$$

in which it denotes the Ekman number $E\left(\equiv \frac{\nu}{\Omega^* L^{*2}}\right)$ ν the kinematic viscosity, Ω^* the basic-state angular velocity and L^* the disk radius.

The scales for dependent variables are acquired from the above basic equations [see, e.g., Greenspan, 1968]:

$$u \sim O(1), \quad v \sim O(1), \quad w \sim O(E^{1/2}), \quad p \sim O(E).$$

It is noted that the radial derivative terms are included in the above equations, which deal with the radial variations of flow due to the presence of axisymmetric surface roughness. The above represent generalized Ekman boundary-layer equations.

The associated boundary conditions are stated: at the disk surface and at the far field :

$$\text{at} \quad z = \delta \cos(2\pi \omega r), \quad u = w = 0, v = r, \quad (5a)$$

$$\text{at} \quad z \to \infty, \quad u = v = 0. \quad (5b)$$

in which $\delta\left[\equiv \frac{a^*}{L^*}\right]$ denotes the nondimensional amplitude and $\omega\left[\equiv\frac{L^*}{\lambda^*}\right]$ the wave number based on disk radius L^*, respectively, of the surface roughness. (The dimensional amplitude and wavelength of the surface roughness are a^* and λ^*, respectively.)

3. Analysis

In the present chapter, a complete formal solution procedure is provided when the system Ekman number has an arbitrary magnitude. An approximate explicit solution will be secured to the second order. Of course, higher all orders of solution can be obtainable along the routine procedure by the method given here.

It is convenient to introduce a variable Ψ by combining eqs.(2) and (3) such that

$$-2i\Psi=E\left(\nabla^2-\frac{1}{r^2}\right)\Psi, \qquad (6)$$

in which $\Psi=v+iu$, i stands for the unit of imaginary number, $i\equiv(-1)^{1/2}$. The boundary conditions, eqs.(5a) & (5b), are expressed as

$$\Psi=r \quad \text{at} \quad z=\delta\cos(2\pi\omega r), \qquad (7a)$$
$$\Psi=0 \quad \text{as} \quad z\rightarrow\infty. \qquad (7b)$$

3.1 Solution for the flat disk problem, $\delta=0$

This reduces to the well-known Ekman layer flow for a flat disk [see, e.g., Greenspan, 1968]. For this case, assuming a solution $\Psi=r\,\Phi_0(z)$ and substituting Ψ into eqs.(6) & (7) produces

$$-2i\,\Phi_0=E\frac{d^2\Phi_0}{dz^2} \qquad (8a)$$

and

$$\Phi_0(0)=1, \quad \Phi_0(z\rightarrow\infty)=0, \qquad (8b)$$

for which the solution is given as

$$\Phi_0=\exp[-(1-i)z/E^{1/2}]. \qquad (9a)$$

Finally, the solution is

$$\Psi[\equiv v+iu]=r\cdot\exp[-(1-i)z/E^{1/2}]. \qquad (9b)$$

3.2 Solution for the wavy disk problem, i.e., $\delta\neq0$

For $\delta\neq0$, it is advantageous to assume the solution to eqs.(6) & (7), from eq.(9b), as

$$\Psi=HI(\xi,\eta), \qquad (10)$$

in which $\xi(\equiv r/E^{1/2})$ and $\eta(\equiv z/E^{1/2})$ are the proper boundary-layer coordinates. The substitution of the form of eq.(10) into eq.(6) leads to

$$-2i\Phi(\xi,\eta)=\left(\nabla^2-\frac{1}{\xi^2}\right)\Phi(\xi,\eta), \qquad (11)$$

in which $\nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}$.

Also, from eqs.(7a) & (7b), the boundary conditions can rewritten as

$$\Phi=1 \quad \text{at} \quad \eta=\alpha\cos(2\pi k\xi), \qquad (12a)$$
$$\Phi=0 \quad \text{at} \quad \eta\rightarrow\infty, \qquad (12b)$$

in which $\alpha=a^*/\left(\frac{\nu}{\Omega^*}\right)^{1/2}$ and $k=\left(\frac{\nu}{\Omega^*}\right)^{1/2}/\lambda^*$. In the rescaling process, it is seen that $\delta\equiv E^{1/2}\alpha$ and $k\equiv E^{1/2}\omega$. Obviously, $(\nu/\Omega^*)^{1/2}$ represents the dimensional thickness of the boundary layer. Therefore, the nondimensional amplitude parameter α indicates the ratio of the roughness amplitude of the disk to the boundary-layer thickness. Similarly, k is the wave number of the disk surface roughness over the length of the boundary-layer thickness.

Assuming that the amplitude parameter α is small but finite, i.e., $\alpha<1$, a series solution with the expansion parameter α is of usefulness :

$$\Phi=\sum_{n=0}^{\infty}\alpha^n\,\Phi_n(\xi,\eta). \qquad (13)$$

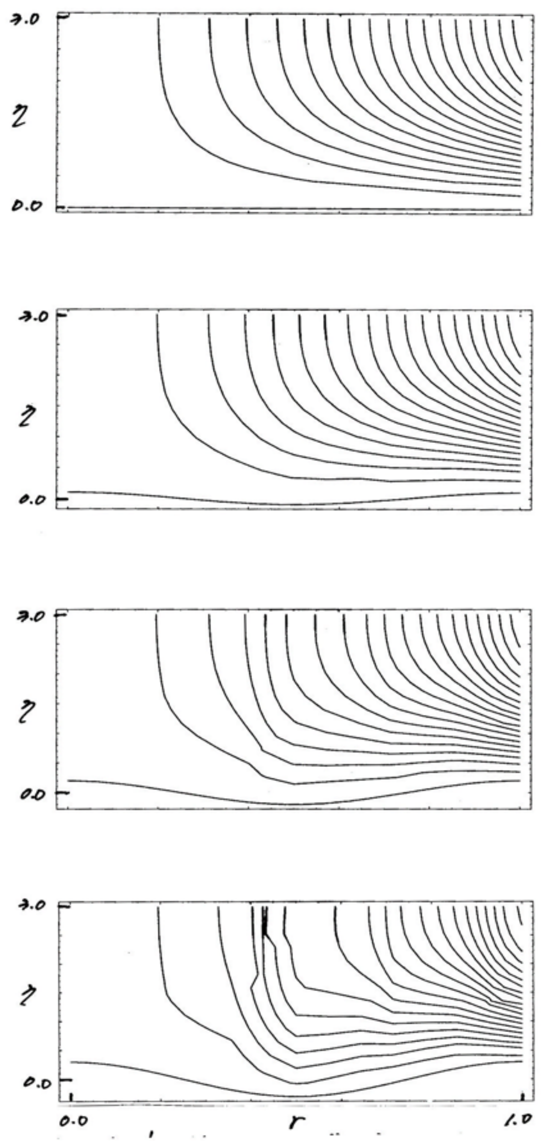

Fig.2 Plots of stream function. $E = 10^{-4}$ and $n = 1$. The amplitudes, a, from the top frame, are : 0, 0.1, 0.2, 0.3.

Clearly, the zeroth-order solution Φ_0 corresponds to the case of a flat-disk, i.e., $\Phi_0 = \exp[-(1-i)\eta]$ from eq.(9).

Placing eq.(13) into eq.(11), one has

$$-2i\,\Phi_n = \left(\nabla^2 - \frac{1}{\xi^2}\right)\Phi_n, (n = 1, 2, \cdots). \quad (14)$$

The boundary condition at the disk surface, by introducing eq.(13) into eq.(12a), generates the relations shown below :

$$\Phi_1(\xi, 0) = (1-i)\cos(2\pi k\xi), \quad (15a)$$

$$\Phi_2(\xi, 0) = i\cos^2(2\pi k\xi) - \frac{\partial\Phi_1(\xi,0)}{\partial\eta}\cos(2\pi k\xi), \quad (15b)$$

$$\cdots\cdot$$

$$\Phi_n(\xi, 0) = \frac{(-1)^{n+1}}{n!}(1-i)^n\cos^n(2\pi k\xi) - \sum_{p=1}^{n-1}\frac{\cos^{n-p}(2\pi k\xi)}{(n-p)!}\frac{d^{n-p}\Phi_p(\xi,0)}{d\eta^{n-p}},$$

$$(n \geq 3). \quad (15c)$$

For the far-field boundary condition, $\eta \to \infty$, one obtains, from eq.(12b) together with eq.(13) :

$$\Phi_n(\xi, \eta \to \infty) = 0, \quad n = 1, 2, 3, \cdots. \quad (15d)$$

Clearly, the above system of equations, eqs.(14) and (15a)-(15d), pose a well-posed eigenvalue problem. By means of the eigenfunction expansion technique, the solution is acquired in the form of the Fourier-Bessel functions.

By way of above procedure, the complete formal solution, under $E \ll 1$ but finite, can now be obtained as

$$\Psi(\equiv v_0 + i\,u_0) = r\sum_{n=0}^{\infty}\alpha^n\,\Phi_n, \quad (16)$$

from which u- and v- solutions to the second-order are given :

$$u_0[\equiv Im\{\Psi\}] = u^{(0)} + \alpha u^{(1)} + \alpha^2 u^{(2)} + O(\alpha^3),$$

$$u^{(0)} = r\,e^{-\eta}\sin(\eta),$$

$$u^{(1)} = r\sum_{n=0}^{\infty}e^{a_n\eta}(\sin(b_n\eta) - \cos(b_n\eta)) \cdot \frac{I_1}{I}J_1(\beta_n r),$$

$$u^{(2)} = r\sum_{n=0}^{\infty}e^{a_n\eta}[(1 + a_n - b_n)\cos(b_n\eta) - (a_n + b_n)\sin(b_n\eta)]\frac{I_2}{I}J_1(\beta_n r),$$

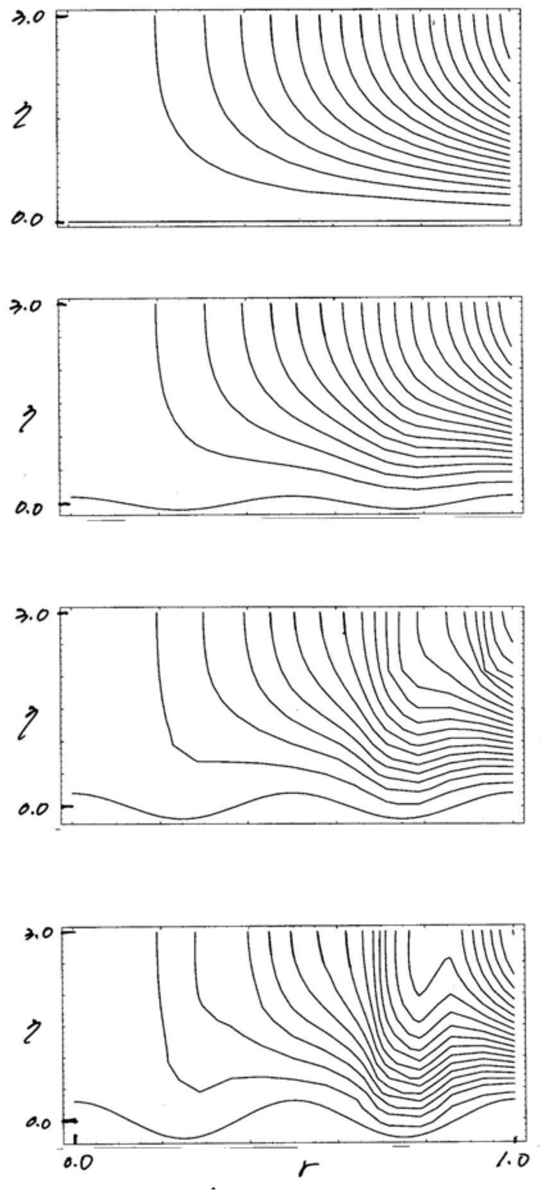

Fig.3 Plots of stream function. $E = 10^{-4}$ and $n = 2$. The amplitudes, a, from the top frame, are : 0, 0.1, 0.2, 0.3.

$$v_0 [\equiv Re\{\Psi\}] = v^{(0)} + \alpha v^{(1)} + \alpha^2 v^{(2)} + O(\alpha^3),$$

$$v^{(0)} = r e^{-\eta} \cos(\eta),$$

$$v^{(1)} = r \sum_{n=0}^{\infty} e^{a_n \eta} (\sin(b_n \eta) + \cos(b_n \eta)) \cdot \frac{I_1}{I} J_1(\beta_n r),$$

$$v^{(2)} = - r \sum_{n=0}^{\infty} e^{a_n \eta} [(1 + a_n - b_n) \sin(b_n \eta) + (a_n + b_n) \cos(b_n \eta)] \frac{I_2}{I} J_1(\beta_n r).$$

It follows that the vertical velocity w and pressure p can be procured by using eqs.(1) & (4).

The actual calculations, by summing up an appropriate number of terms in the above equations, are straightforward and easy to implement.

Exemplary stream functions near the wavy disk are ploted Figs.2-3.

In order to gain further insight into the basic physics, an asymptotic analysis under $E \rightarrow 0$ is productive. In the next chapter, an explicit complete solution will be examined.

4. Asymptotic solution, $E \rightarrow 0$

In an effort to reinforce the complete formal solution of the previous chapters, the asymptotic solution in the limit $E \rightarrow 0$ will be discussed.

The radial velocity, u, is

$$u = r \cdot Im \left\{ \sum_{n=0}^{\infty} \alpha^n \Phi_n \right\}$$

$$= r e^{-\eta} [\sin(\eta) + \sum_{n=1}^{\infty} (\sqrt{2}\alpha)^n C_n \sin\left(\eta - \frac{\pi n}{4}\right) \cos^n(2\pi\omega r)] \quad (17a)$$

The azimuthal velocity field, v, is

$$v = r \cdot Re \left\{ \sum_{n=0}^{\infty} \alpha^n \Phi_n \right\}$$

$$= r e^{-\eta} [\cos(\eta) + \sum_{n=1}^{\infty} (\sqrt{2}\alpha)^n C_n \cos\left(\eta - \frac{\pi n}{4}\right) \cos^n(2\pi\omega r)] \quad (17b)$$

The vertical velocity field, w, is obtained from eq.(1) :

$$w = - \int \frac{1}{r} \frac{\partial(ru)}{\partial r} d\eta$$

$$= \sqrt{2} e^{-\eta} \left\{ \sin\left(\eta + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} (\sqrt{2}\alpha)^n C_n \sin\left(\eta + \frac{(1-n)\pi}{4}\right) \cdot [\cos^n(2\pi\omega r) - (\pi\omega n) r \cos^{n-1}(2\pi\omega r) \sin(2\pi\omega r)] \right\} - 1. \quad (17c)$$

The last term in eq.(17c), $W_\infty(r)$, is an integration constant. This can be determined by making use of the non-permeable condition at the disk surface. For this purpose, the non-permeable condition, i.e., $w(r,\eta=\alpha\cos(2\pi\omega r))=0$, is Taylor-expanded at $\eta=0$. The resulting equation at each order is considered for small α, which leads to the determination of the integration constant, i.e., $W_\infty(r)=-1$.

It is seen that, as shown in eq.(17c), as $\eta\rightarrow\infty$, $w\rightarrow-1$. This implies that the Ekman layer pumping velocity at the boundary layer edge remains unaffected by the introduction of the wall waviness. The non-flatness of the disk has influence on the vertical velocity distribution only within the boundary layer interior.

It is useful to define the horizontally-averaged velocity at $\eta\geq\alpha$ as

$$<X(\eta,r)>=\int_0^1 X\,dr,$$

in which X stands for u or v. In view of eqs.(17a) & (17b), the averaged u- and v-velocities are

$$<u>=\frac{1}{2}e^{-\eta}\sin(\eta)-\frac{1}{4}\alpha^2 e^{-\eta}\cos(\eta)+O(\alpha^3),$$

$$<v>=\frac{1}{2}e^{-\eta}\cos(\eta)+\frac{1}{4}\alpha^2 e^{-\eta}\sin(\eta)+O(\alpha^3).$$

In the above calculation, for simplicity, the wave number ω is assumed to be integer. As expected, the first term is the same for the case of a flat disk and the effect of waviness reflects on the second term.

The physical meaning of the above expressions is interesting. For a disk with surface roughness with amplitude a, in compression to the case of a flat disk, the azimuthal velocity increases in magnitude, together with the thickening boundary layer. The radial velocity u, however, decreases in magnitude as a increases with η fixed. Consequently, the overall Ekman pumping at the edge of the boundary layer remains unchanged, maintaining the constant value given in eq.(17c). In summary, within the framework of the stated assumption of the present analysis, the Ekman pumping is unaffected by the presence of the surface roughness of the disk.

5. Conclusions

An Ekman boundary layer flow induced by rapidly-rotating wavy disk with sinusoidal axisymmetric topography has been studied. The system Ekman number is assumed to be very small, i.e., $E\ll 1$. The amplitude of disk waviness is assumed to be order of the thickness of Ekman boundary layer, i.e., $\delta\sim(E^{1/2})$ and the wave number of the disk surface is assumed to be very large, i.e., $n\leq O(E^{-1/2})$, which means the radial diffusion term is negligible no longer.

An analytic solution has been obtained for $E\ll 1$ but finite. Also, asymptotic solution, when $E\rightarrow 0$, was secured. These solutions clearly show some physics of boundary layer flows with surface waviness-effect. For a disk with surface roughness, in contrast to the case of a flat disk, the azimuthal velocity increases in magnitude, together with the thickening boundary layer. The radial velocity, however, decreases in magnitude as the amplitude of surface waviness increases. Consequently, the overall Ekman pumping at the edge of the boundary layer remains unchanged, maintaining the constant value, which is equal to that of the flat disk.

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