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Intuitionistic Fuzzy Metric Spaces

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Abstract

Using the idea of intuitionistic fuzzy set due to Atanassov, we define the notion of intuitionistic fuzzy metric spaces as a natural generalization of fuzzy metric spaces due to George and Veeramani and prove some known results of metric spaces including Baire's theorem and the Uniform limit theorem for intuitionistic fuzzy metric spaces.

1. Intuitionistic fuzzy metric spaces

One of the most important problems in Fuzzy Topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of views. In particular, George and Veeramani [12] have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek [17]. On the other hand, Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets, and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [1-7,11,13].

In this paper, using the idea of intuitionistic fuzzy sets, we define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [12]. In section 3, we define a Hausdorff topology on this intuitionistic fuzzy metric space and show that every metric induces an intuitionistic fuzzy metric. Further we introduce the notion of Cauchy sequences in an intuitionistic fuzzy metric space and prove the Baire's theorem [21] for intuitionistic fuzzy metric spaces. In section 4, we are finding a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete and show that every separable intuitionistic fuzzy metric space is second countable and that every subspace of an intuitionistic fuzzy metric space is separable. Finally, we prove the Uniform limit theorem [20] for intuitionistic fuzzy metric spaces.

Definition 1.1 A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$

(a) $M(x,y,t)+N(x,y,t)\le 1$;
(b) $M(x,y,t)>0$;
(c) $M(x,y,t)=1$ if and only if $x=y$;
(d) $M(x,y,t)=M(y,x,t)$;
(e) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;
(f) $M(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous;
(g) $N(x,y,t)>0$;
(h) $N(x,y,t)=0$ if and only if $x=y$;
(i) $N(x,y,t)=N(y,x,t)$;
(j) $N(x,y,t)\diamond N(y,z,s)\ge N(x,z,t+s)$;
(k) $N(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Example 1.2 (Induced intuitionistic fuzzy metric) Let (X,d) be a metric space. Denote $a*b=ab$ and $a\diamond b=\min\{1,a+b\}$ for all $a,b\in[0,1]$ and let M_d and N_d be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$M_{d(x,y,t)}=\frac{ht^n}{ht^n+md(x,y)},\quad\text{and}$$
$$N_{d(x,y,t)}=\frac{d(x,y)}{kt^n+m\,d(x,y)}$$

for all $h,k,m,n\in R^+$. Then $(X,M_d,N_d,*,\diamond)$ is an intuitionistic fuzzy metric space.

Remark 1.3 Note the above example holds even with the t-norm $a*b=\min\{a,b\}$ and the t-conorm $a\diamond b=\max\{a,b\}$ and hence (M,N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h=k=m=n=1$, we get $M_d(x,y,t)=\frac{t}{t+d(x,y)}$, and $N_d(x,y,t)=\frac{d(x,y)}{t+d(x,y)}$. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 1.4 Let $X=N$. Define $a*b=\max\{0,a+b-1\}$ and $a\diamond b=a+b-ab$ for all $a,b\in[0,1]$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$ as follows:

$$M(x,y,t)=\begin{cases}\frac{x}{y} & \text{if } x\le y,\\ \frac{y}{x} & \text{if } y\le x,\end{cases}\quad\text{and}$$

$$N(x,y,t)=\begin{cases}\frac{y-x}{y} & \text{if } x\le y,\\ \frac{x-y}{x} & \text{if } y\le x,\end{cases}$$

for all $x,y\in X$ and $t>0$. Then $(X,M,N,*,\diamond)$ is an intuitionistic fuzzy metric space.

Remark 1.5 Note that, in the above example, t-norm $*$ and t-conorm $\diamond$ are not associated. And there exists no metric d on X satisfying

$$M(x,y,t)=\frac{t}{t+d(x,y)},\quad\text{and}$$

$$N(x,y,t)=\frac{d(x,y)}{t+d(x,y)}$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note that the above functions (M,N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a*b=\min\{a,b\}$ and $a\diamond b=\max\{a,b\}$.

2. Topology induced by an intuitionistic fuzzy metric

Defintion 2.1 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space, and let $r\in(0,1)$, $t>0$ and $x\in X$. The set $B(x,r,t)=\{y\in X: M(x,y,t)>1-r,\ N(x,y,t)<r\}$ is called the open ball with center x and radius r with respect to t.

Theorem 2.2 Every open ball $B(x,r,t)$ is an open set.

Remark 2.3 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. Define

$\tau_{(M,N)}=\{A\subset X:$ for each $x\in A$, $\exists\, t>0$ and $r\in(0,1)$ s.t. $B(x,y,t)\subset A\}$.

Then $\tau_{(M,N)}$ is a topology on X.

Remark 2.4 (a) From Theorem 2.2 and Remark 2.3, every intuitionistic fuzzy metric (N,M) on X generates a topology $\tau_{(M,N)}$ on X which has as a base the family of open sets of the form $\{B(x,r,t): x\in X,\ r\in(0,1),\ t>0\}$.

(b) Since $\{B(x,\frac{1}{n},\frac{1}{n}): n=1,2,\cdots\}$ is a local base at x, the topology $\tau_{(M,N)}$ is first countable.

Theorem 2.5 Every intuitionistic fuzzy metric space is Hausdorff.

Remark 2.6 Let (X,d) be a metric space. Let
$M(x,y,t)=\frac{t}{t+d(x,y)}$, $N(x,y,t)=\frac{d(x,y)}{kt+d(x,y)}$,
where $k\in R^{+}$,
be the intuitionistic fuzzy metric defined on X. Then the topology τ_d induced by the metric d and the topology $\tau_{(M,N)}$ induced by the intuitionistic fuzzy metric (N,M) are the same.

Definition 2.7 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. A subset A of X is said to be IF-bounded if there exist $t>0$ and $r\in(0,1)$ such that
$M(x,y,t)>1-r$ and $N(x,y,t)<r$ for all $x,y\in A$.

Remark 2.8 Let $(X,M,N,*,\diamond)$be an intuitionistic fuzzy metric space induced by a metric d on X. Then $A\subset X$ is IF-bounded if and only if it is bounded.

Theorem 2.9 Every compact subset A of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is IF-bounded.

Remark 2.10 In an intuitionistic fuzzy metric space every compact set is closed and bounded.

Theorem 2.11 Let $(X,M,N,*,\diamond)$ be a fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n\to x$ if and only if $M(x_n,x,t)\to 1$ and $N(x_n,x,t)\to 0$ as $n\to\infty$.

Theorem 2.16 A subset A of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is nowhere dense iff every nonempty open set in X contains an open ball whose closure is disjoint from A.

Theorem 2.17 (Baire's Theorem)
Let $\{U_n : n\in \mathrm{N}\}$ be a sequence of dense open subsets of a complete intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$. Then $\bigcap_{n\in \mathrm{N}} U_n$ is also dense in X.

Note: Since any complete intuitionistic fuzzy metric space cannot be represented as the union of a sequence of nowhere dense sets, it is not of the first category. Hence every complete intuitionistic fuzzy metric space is of the second category.

Remark 2.18 Since every metric induces an intuitionistic fuzzy metric and intuitionistic fuzzy metric is a generalization of fuzzy metric, Baire's Theorem for complete metric space and Baire's Theorem for complete fuzzy metric space are particular cases of the above theorem.

3. Some properties of complete intuitionistic fuzzy metric spaces

Definition 3.1 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. A ollection $\{F_n\}_{n\in N}$ is said to have intuitionistic fuzzy diameter zero if for each $r\in(0,1)$ and each $t>0$, there exists $n_0\in \mathrm{N}$ such that $M(x,y,t)>1-r$ and $N(x,y,t)<r$ for all $x,y\in F_{n_0}$.

Remark 3.2 A nonempty subset F of an intuitionistic fuzzy metric space X has intuitionistic fuzzy diameter zero if and only if F is a singleton set.

Theorem 3.3 An intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is complete if and only if every nested sequence $\{F_n\}_{n\in N}$ of nonempty closed sets with intuitionistic fuzzy diameter zero have nonempty intersection.

Remark 3.4 The element $x\in\bigcap_{n\in \mathrm{N}} F_n$ is unique. For if there are two elements $x,y\in\bigcap_{n\in \mathrm{N}} F_n$, since $\{F_n\}_{n\in N}$ has intuitionistic fuzzy diameter zero, for each fixed $t>0$, $M(x,y,t)>1-1/n$ and $N(x,y,t)<1/n$ for each $n\in \mathrm{N}$. This implies $M(x,y,t)=1$ and $N(x,y,t)=0$ and hence $x=y$.

Note that the topologies induced by the standard intuitionistic fuzzy metric and the corresponding metric are the same. So we have the following:

Corollary 3.5 A metric space (X,d) is complete if and only if every nested sequence $\{F_n\}_{n\in N}$ of nonempty closed sets with diameter tending to zero have non- empty intersection.

Theorem 3.6 Every separable intuitionistic fuzzy metric space is second countable.

Remark 3.7 Since second countability is hereditary property and second countability implies separability, we obtain the following: Every subspace of a separable intuitionistic fuzzy metric space is separable.

Definition 3.8 Let X be any nonempty set and $(Y, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to converge uniformly to a function f from X to Y if given $t>0$ and $r \in (0,1)$, there exists $n_0 \in \mathrm{N}$ such that $M(f_n(x), f_n(y), t) > 1-r$ and $N(f_n(x), f_n(y), t) < r$ for all $n \geq n_0$ and for all $x \in X$.

Theorem 3.9 (Uniform Limit Theorem)
Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to an intuitionistic fuzzy metric space $(Y, M, N, *, \diamond)$. If $\{f_n\}$ converges uniformly to $f: X \to Y$, then f is continuous.

6. 참고문헌

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