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Intuitionistic Fuzzy Normal Subgroups and Intuitionistic Fuzzy Cosets

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Abstract

We study some properties of intuitionistic fuzzy normal subgroups of a group. In particular, we obtain two characterizations of intuitionistic fuzzy normal subgroups. Moreover, we introduce the concept of an intuitionistic fuzzy coset and obtain several results which are analogs of some basic theorems of group theory.

Key words and phrases : intuitionistic fuzzy normal subgroup, intuitionistic fuzzy coset, intuitionistic fuzzy quotient group.

0. Introduction

The concept of a fuzzy set was introduced by Zadeh in [16], and since then these has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behavior studies. In particular, several researchers [6, 13, 14, 15] applied the notion of a fuzzy set to group theory.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues [4,5,7], and Lee and Lee[12] applied the notion of intuitionistic fuzzy sets to topology. Also, several researchers [2,3,9,10,11] applied one to algebra.

In this paper, we investigate some properties of intuitionistic fuzzy normal subgroups of a group. In particular, we obtain two characterizations of intuitionistic fuzzy normal subgroups. Moreover, we introduce the concept of intuitionistic fuzzy cosets and obtain several results which are analogs of some basic theorems of group theory.

1. Preliminaries

We will list some concepts and results

needed in the later sections.

For sets X, Y and Z, $f=(f_1,f_2):X\to Y\times Z$ is called a *complex mapping* if $f_1:X\to Y$ and $f_2:X\to Z$ are mappings.

Throughout this paper, we will denote the unit interval [0,1] as I.

Definition 1.1[1]. Let X be a nonempty set. A complex mapping $A=(\mu_A,\nu_A):X\to I\times I$ is called an *intuitionistic fuzzy set* (in short, IFS) on X if $\mu_A+\nu_A\leq 1$, where the mapping $\mu_A:X\to I$ and $\nu_A:X\to I$ denote the degree of membership (namely $\mu_A (x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x\in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[1]. Let X be a nonempty set and let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be IFSs on X. Then

(1) $A\subset B$ iff $\mu_A\leq\mu_B$ and $\nu_A\geq\nu_B$.
(2) $A=B$ iff $A\subset B$ and $B\subset A$.
(3) $A^c=(\nu_A,\mu_A)$.
(4) $A\cap B=(\mu_A\wedge\mu_B,\nu_A\vee\nu_B)$.
(5) $A\cup B=(\mu_A\vee\mu_B,\nu_A\wedge\nu_B)$.
(6) $[\ ]A=(\mu_A,1-\mu_A), \langle\ \rangle A=(1-\nu_A,\nu_A)$.

Definition 1.3[1] Let $\{A_i\}_{i\in J}$ be an arbitrary family of IFSs in X, where $A_i=(\mu_{A_i},\nu_{A_i})$ for each $i\in J$. Then

(a) $\bigcap A_i=(\bigwedge\mu_{A_i},\bigvee\nu_{A_i})$.
(b) $\bigcup A_i=(\bigvee\mu_{A_i},\bigwedge\nu_{A_i})$.

Definition 1.4[4]. $0_\sim=(0,1)$ and $1_\sim=(1,0)$.

Definition 1.5[4]. Let X and Y be nonempty sets and let $f:X\to Y$ be a mapping. Let $A=(\mu_A,\nu_A)$ be an IFS in X and $B=(\mu_B,\nu_B)$ be an IFS in Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:
$$f^{-1}(B)=(f^{-1}(\mu_B),f^{-1}(\nu_B)),$$
where $f^{-1}(\mu_B)=\mu_B\circ f$ and $f^{-1}(\nu_B)=\nu_B\circ f$.
(b) the image of A under f, denoted by $f(A)$, is the IFS in Y defined by:
$$f(A)=(f(\mu_A),f(\nu_A)),$$
where for each $y\in Y$
$$f(\mu_A)(y)=\begin{cases}\bigvee_{x\in f^{-1}(y)}\mu_A(x) & \text{if } f^{-1}(y)\neq\emptyset,\\ 0 & \text{if } f^{-1}(y)=\emptyset\end{cases}$$
and
$$f(\nu_A)(y)=\begin{cases}\bigwedge_{x\in f^{-1}(y)}\nu_A(x) & \text{if } f^{-1}(y)\neq\emptyset,\\ 0 & \text{if } f^{-1}(y)=\emptyset.\end{cases}$$

2. Intuitionistic fuzzy normal subgroups

We first prove two Lemmas before introducing the definition of an intuitionistic fuzzy normal subgroup.

Lemma 2.1. If A is an IFGP of a finite group G, then A is an IFG of G.

Lemma 2.2. Let A be an IFG of a group G and let $x\in G$. Then $A(xy)=A(y)$ for each $y\in G$ if and only if $A(x)=A(e)$.

Remark 2.3. It is easy to see that if $A(x)=A(e)$, then $A(xy)=A(yx)$ for each $y\in G$.

Definition 2.4[11]. Let A be an IFG of a group G. Then A is called an *intuitionistic fuzzy normal subgroup* (in short, IFNG) if $A(xy)=A(yx)$ for any $x,y\in G$.

We will denote the set of all IFNGs of G as IFNG(G).

Let G be a group and $a,b\in G$. We say that a is *conjugate* to b if there exists $x\in G$ such that $b=x^{-1}ax$. It is well-known that conjugacy is an equivalence relation on G. The equivalence classes in G under the relation of conjugacy are called *conjugate classes* [8].

Theorem 2.5. Let A be an IFG of a group G. Then $A \in IFNG(G)$ if and only if A is constant on the conjugate classes of G.

The following is the generalization of the above result using intuitionistic fuzzy sets.

Theorem 2.6. Let $A \in IFG(G)$. Then $A \in IFNG(G)$ if and only if $\mu_A([x,y]) \geq \mu_A(x)$ and $\nu_A([x,y]) \leq \nu_A(x)$ for any $x, y \in G$.

Theorem 2.7. Let $A \in IFNG(G)$ and let $(\lambda, \mu) \in I \times I$ such that $\$\lambda \leq \mu_A(e)$, $\mu \geq \nu_A(e)$ and $\lambda + \mu \leq 1$, where e denotes the identity of G. Then . $A^{(\lambda,\mu)} \lhd G$.

The following is the immediate result of Theorem 2.7:

Corollary 2.7[11, Proposition 3.5]. Let A be an IFNG of a group G with identity e. Then $G_A \lhd G$, where $G_A = \{x \in G : A(x) = A(e)\}$.

The following is the converse of Theorem 2.7:

Theorem 2.8. If A is an IFG of a finite group G such that all the level subgroups of A are normal in G, then $A \in IFNG(G)$.

Example 2.9. Let G be the group of all symmetries of a square. Then G is a group of order 8 generated by a rotation through $\pi/2$ and a reflection along a diagonal of the square. Let us denote the elements of G by $\{1,2,3,4,5,6,7,8\}$, where 1 is the identity, 2 is rotation through $\pi/2$, and 5 is a reflection along a diagonal: the multiplication table of G is as shown in Table 1.

We can easily see that the conjugate classes

	1	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8
	2	3	4	1	6	7	8	5
	3	4	1	2	7	8	5	6
	4	1	2	3	8	5	6	7
	5	8	7	6	1	4	3	2
	6	5	8	7	2	1	4	3
	7	6	5	8	3	2	1	4
	8	7	6	5	4	3	2	1

of G are $\{1\}, \{3\}, \{5,7\}, \{6,8\}, \{2,4\}$.

Table 1.

Let $H = \{1,3\}$ and let $K = \{1,2,3,4\}$. Then clearly, $H \lhd G$ and $K \lhd G$ (in fact, H is the center of G. Thus we have a chain of normal subgroups given by

$$1 \subset H \subset K \subset G. \qquad (**)$$

Now we will construct an IFG of G whose level subgroups are precisely the members of the chain (**). Let $(t_i, s_i) \in I \times I$, $0 \leq i \leq 3$ such that $t_i + s_i \leq 1$, $t_0 > t_1 > t_2 > t_3$ and $s_0 < s_1 < s_2 < s_3$.

Define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows:

$$A(1) = (t_0, s_0),\ A(H \setminus \{1\}) = (t_1, s_1),$$
$$A(K \setminus H) = (t_2, s_2),\ A(G \setminus K) = (t_3, s_3).$$

From the definition of A, it is clear that $A(x) = A(x^{-1})$ for each $x \in G$. Also, we can easily check that for any $x, y \in G$

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

Furthermore, it is clear that A is constant on the conjugate classes. Hence, by Theorem 2.5, $A \in IFNG(G)$.

Remark 2.10. Example 2.9 is the generalization of Example 3.10 in [14] using intuitionistic fuzzy sets.

Theorem 2.11. Let $f: G \to G'$ be a group homomorphism. If A is an IFNG of $f(G)$, then $f^{-1}(A) \in IFNG(G)$.

The following results leads to the motivation behind the definition of an intuitionistic fuzzy coset in the next section.

Theorem 2.12. Let A be an IFNG of a group G with identity e. We define a complex mapping $\mathcal{A}=(\mu_{\mathcal{A}}, \nu_{\mathcal{A}}): G/G_A \to I \times I$ as follows : for each $x \in G$,

$$\mathcal{A}(G_A x)=A(x).$$

Then $\mathcal{A} \in IFNG(G/G_A)$. Conversely, if $N \lhd G$ and $\mathcal{B} \in IFNG(G/N)$ such that $\mathcal{B}(Ng)=\mathcal{B}(N)$ only when $g \in N$, then there exists an $A \in IFNG(G)$ such that $G_A=N$ and $\mathcal{A}=\mathcal{B}$.

3. Intuitionistic fuzzy cosets and fuzzy Lagrange's Theorem

Definition 3.1. Let A be an IFG of a group G and let $x \in G$. We define a complex mapping $Ax=(\mu_{Ax}, \nu_{Ax}): G \to I \times I$ are follows : for each $g \in G$,

$$Ax(g)=A(gx^{-1}), \text{ i.e.,}$$

$$\mu_{Ax}(g)=\mu_A(gx^{-1}) \text{ and } \nu_{Ax}(g)=\nu_A(gx^{-1}).$$

Then A is called the *intuitionistic fuzzy coset* of G determined by x and A.

Remark 3.2. Definition 3.1 extends in a natural way the usual definition of a "coset" of a group. This is seen as follows : Let H be a subgroup of a group G and let $A=(\chi_H, \chi_{H^c})$, where χ_H is the characteristic function of H. Let $x, g \in G$. Then $Ax=(\chi_{Hx}, \chi_{H^c x})$.

Proposition 3.3. Let A be an IFNG of a group G and let $x \in G$. Then $Ax(xg)=Ax(gx)=A(g)$ for each $g \in G$.

Remark 3.4. Proposition 3.3 is analogous to the result in group theory that if $N \lhd G$, then $Nx=xN$ for each $x \in G$.

If N is a normal subgroup of a group G, then the cosets of G with respect to N form a group (called the *quotient group* G/N). For an IFNG, we have the analoguous result.

Theorem 3.5. Let A be an IFNG of a group G and let G/A be the set of all the intuitionistic fuzzy cosets of A. We define an operation $*$ on G/A as follows : for any $x, y \in G$,

$$Ax * Ay = Axy.$$

Then $(G/A, *)$ is a group.

Proposition 3.6. Let A be an IFNG of a group G. We define a complex mapping $\overline{A}=(\mu_{\overline{A}}, \nu_{\overline{A}}): G/A \to I \times I$ as follows : for each $x \in G$,

$$\overline{A}(Ax)=A(x), \text{ i.e.,}$$

$$\mu_{\overline{A}}(Ax)=\mu_A(x) \text{ and } \nu_{\overline{A}}(Ax)=\nu_A(x).$$

Then $\overline{A}$ is an IFG of G/A. In this case, $\overline{A}$ is called the *intuitionistic fuzzy quotient group* determined by A.

Proposition 3.7. Let A be an IFNG of a group G. We define a mapping $\theta: G \to G/A$ as follows : for each $x \in G$, $\theta(x)=Ax$. Then θ is a homomorphism with $\mathrm{Ker}(\theta)=G_A$.

Now we obtain for an IFG an analogous result of the "Fundmental Theorem of Homomorphism of Groups".

Theorem 3.8. Let $A \in IFNG(G)$. Then each intuitionistic fuzzy (normal) subgroup of G/A corresponds in a natural way to an intuitionistic fuzzy (normal) subgroup of G.

Now we will obtain an intuitionistic fuzzy analog of the famous "Lagrange's Theorem" for finite groups which is a basic result in group theory . Let A be an IFG of a finite group G. Then it clear that G/A is finite.

Definition 3.9. Let A be an IFG of a finite group G. Then the cardinality $|G/A|$ of G/A is called the *index* of A.

Theorem 3.10[Intuitionistic Lagrange's

Theorem]. Let A be an IFG of a finite group G. Then the index of A divides the order of G.

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