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Intuitionistic Fuzzy Ideals on A Distributive Lattice

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Abstract

We introduce the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy congruences on a lattice, and discuss the relationship between intuitionistic fuzzy ideals and intuitionistic fuzzy congruence on a distributive lattice. Also we prove that for a generalized Boolean algebra, the lattice of intuitionistic fuzzy ideals is isomorphic to the lattice of intuitionistic fuzzy congruences. Finally, we consider the products of intuitionistic fuzzy ideals and obtain a necessary and sufficient condition for an intuitionistic fuzzy ideals on the direct sum of lattices to be representable on a direct sum of intuitionistic fuzzy ideals on each lattice.

Key words and phrases : intuitionistic fuzzy filter, intuitionistic fuzzy ideal, intuitionistic fuzzy congruence.

0. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [20], several researchers [9,15,17,18] have applied the notion of fuzzy sets to group theory. In particular, Yuan and Wu [19] have applied one to lattice theory.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues [7,8,11], and Lee and Lee[16] applied the notion of intuitionistic fuzzy sets to topology. Another researchers [3,4,12,13] applied one to group theory. In particular, Burillo and Bastince [5,6], and Deschrijver and E. E. Kerre[10] introduced the concept of intuitionistic fuzzy relations and investigated some of it's properties.

In this paper, we introduce the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy congruences on a lattice, and discuss the relationship between intuitionistic fuzzy ideals and intuitionistic fuzzy congruence on a distributive lattice. Also we prove that for a generalized Boolean algebra, the lattice of intuitionistic fuzzy ideals is isomorphic to the lattice of intuitionistic fuzzy congruences. Finally, we consider the products of intuitionistic fuzzy ideals and obtain a necessary and sufficient condition for an intuitionistic fuzzy ideals on the direct sum of lattices to be representable on a direct sum of intuitionistic fuzzy ideals on each lattice.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f=(f_1, f_2): X \to Y \times Z$ is called a *complex mapping* if $f_1: X \to Y$ and $f_2: X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I.

Definition 1.1[2]. Let X be a nonempty set. A complex mapping $A=(\mu_A, \nu_A): X \to I \times I$ is called an intuitionistic fuzzy set (in short, *IFS*) on X if $\mu_A + \nu_A \leq 1$, where the mapping $\mu_A: X \to I$ and $\nu_A: X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[2]. Let X be a nonempty set and let $A=(\mu_A, \nu_A)$ and $B=(\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1-\mu_A)$, $\langle\rangle A = (1-\nu_A, \nu_A)$.

Definition 1.3[5]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[5]. $0_\sim = (0,1)$ and $1_\sim = (1,0)$.

Definition 1.5[5]. Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping. Let $A=(\mu_A, \nu_A)$ be an IFS in X and $B=(\mu_B, \nu_B)$ be an IFS in Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$ and $f^{-1}(\nu_B) = \nu_B \circ f$.

(b) the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Now, we list some concepts and results in lattice theory.

Throughout this paper $L=(L,+,\cdot)$ denotes a lattice.

Definition 1.6[2]. An element $x\in L$ is said to be relatively complemented if x is complemented in every $[a,b]$ with $a\leq x\leq b$, i.e., $x+y=b$ for some $y\in[a,b]$ such that $xy=a$. The lattice L is said to be relatively complemented if each $x\in L$ is relatively complemented.

Definition 1.7[2]. A relatively complemented distributive lattice with 0 is a generalized Boolean algebra

Definition 1.8[2]. Let L be a generalized Boolean algebra and let $x,y\in L$ We define the difference, $x-y$ and the symmetric difference, $x\oplus y$, of x and y, respectively as follows :

$x-y$ is the relative complement of xy in the interval $[0,x]$

and

$x\oplus y=(x-y)+(y-x)$.

It is easily seen that :

(1) $x-y\leq x$.

(2) $y+(x-y)=x+y$.

(3) $y(x-y)=0$.

Result 1.A. [19, Lemma 3.3]. Let L be a generalized Boolean algebra. Then $x+y=x\oplus y\oplus xy$ for any $x,y\in L$.

Definition 1.9[14]. A ring with 1 in which every element is idempotent is called a Boolean ring.

Result 1.B[14, Lemma 1.9]. Let R be a Boolean ring. Then

(1) R is commutative.

(2) $a+a=0$ for each $a\in R$.

Result 1.C[2, Exercise 4 in p. 55].

(1) If $(L,+,\cdot,0)$ is a generalized Boolean algebra, then $(L,\oplus,\cdot,0)$ is a Boolean ring.

(2) If $(R,\oplus,\cdot,0)$ is a Boolean ring, then $(R,+,\cdot,0)$ is a generalized Boolean algebra, where $x+y=x\oplus y\oplus xy$. Moreover, $x\oplus y=(x-y)+(y-x)$.

The following is the immediate result of Definition 1.9, Results 1.A, 1.B and 1.C:

Lemma 1.10. Let L be a generalized Boolean algebra. Then

$$x+(x\oplus,y)=y+(x\oplus y)\text{ for any } x,y\in L.$$

Remark 1.10. We can see that Lemma 1.9 is proved in Lemma 3.4 in [7, 19].

2. Intuitionistic fuzzy sublattices, ideals and filters

Definition 2.1. Let $A\in IFS(L)$. (1) A is called an *intuitionistic fuzzy sublattices* (in short, *IFL*) of L if it satisfies the following corditions : for any $x,y\in L$,

$$\mu_A(x+y)\wedge\mu_A(xy)\geq\mu_A(x)\wedge\mu_A(y)$$

and

$$\nu_A(x+y)\vee\nu_A(xy)\leq\nu_A(x)\vee\nu_A(y).$$

(2) A is called an *intuitionistic fuzzy filter* (in short, *IFF*) of L,

if (i) A is an *IFG* of L,

(ii) A is monotonic, i.e., $\mu_A(x)\leq\mu_A(y)$ and $\nu_A(x)\geq\nu_A(y)$ whenever $x\leq y$.

(3) A is called an intuitionistic fuzzy ideal} (in sort, *IFI*) of L

if (i) A is an *IFL* of L

(ii) A is antimonotonic, i.e., $\mu_A(x)\geq\mu_A(y)$ and $\nu_A(x)\leq\nu_A(y)$ whenever $x\leq y$.

We will denote the set of all *IFLs*, *IFFs* and *IFIs* of L as $IFL(L)$, $IFF(L)$ as $IFI(L)$, respectively.

Proposition 2.2. Let $A\in IFL(L)$. Then

(1) $A\in IFF(L)$ if and only if $A(xy)=(\mu_A(x)\wedge\mu_A(y),\nu_A(x)\vee\nu_A(y))$ for any $x,y\in L$, i.e., $\mu_A:(L,\cdot)\to(I,\wedge)$ and

$\nu_A:(L,\cdot)\to(I,\vee)$ are homomorphisms.

(2) $A\in IFI(L)$ if and only if

$$A(x+y)=(\mu_A(x)\wedge\mu_A(y),\nu_A(x)\vee\nu_A(y))$$

for any $x,y\in L$, i.e., $\mu_A:(L,+)\to(I,\wedge)$ and $\nu_A:(L,+)\to(I,\vee)$ are homomorphisms.

Definition 2.3. An intuitionistic fuzzy filter [resp. ideal] A of L is said to be prime if for any $x,y\in L$,

$$\mu_A(x+y)\le\mu_A(x)\vee\mu_A(y)$$

and

$$\nu_A(x+y)\ge\nu_A(x)\wedge\nu_A(y)$$

[resp. $\mu_A(xy)\le\mu_A(x)\vee\mu_A(y)$ and $\nu_A(xy)\ge\nu_A(x)\wedge\nu_A(y)$].

Proposition 2.4. Let $A\in IFL(L)$.

(1) A is an intuitionistic fuzzy prime filter of L if and only if

$$A(xy)=(\mu_A(x)\wedge\mu_A(y),\nu_A(x)\vee\nu_A(y))$$

and

$$A(x+y)=(\mu_A(x)\wedge\mu_A(y),\nu_A(x)\vee\nu_A(y))$$

for any $x,y\in L$, i.e., $\mu_A:(L,+,\cdot)\to(I,\vee,\wedge)$ and $\nu_A:(L,+,\cdot)\to(I,\wedge,\vee)$ are homomorphisms.

(2) A is an intuitionistic fuzzy prime filter of L if and only if

$$A(x+y)=(\mu_A(x)\wedge\mu_A(y),\nu_A(x)\vee\nu_A(y))$$

and

$$A(xy)=\mu_A(x)\wedge\mu_A(y),\nu_A(x)\vee\nu_A(y))$$

for any $x,y\in L$, i.e., $\mu_A:(L,+,\cdot)\to(I,\wedge,\vee)$ and $\nu_A:(L,+,\cdot)\to(I,\vee,\wedge)$ are homomorphisms.

The following is the immediate result of Proposition 2.4 :

Corollary 2.4. Let $A\in IFL(L)$. Then A is an intuitionistic fuzzy prime filter (resp. ideal) of L if and only if A^c is an intuitionistic fuzzy prime ideal (resp. filter) of L.

Proposition 2.5. Let $f:L\to L'$ be a lattice homomorphism.

(1) If f is surjective and $A\in IFL(L)$ [resp. $IFI(L)$ and $IFF(L)$], then $f(A)\in IFL(L')$ [resp. $IFI(L')$ and $IFF(L')$].

(2) If B is an intuitionistic fuzzy sublattice [resp. ideal, prime ideal, filter, and prime filter] of L', then $f^{-1}(B)$ is an intuitionistic fuzzy sublattice [resp. ideal, prime ideal, filter, and prime filter] of L.

3. Intuitionistic fuzzy congruences

Definition 3.1. Let $R\in IFR(L)$. Then R is called an *intuitionistic fuzzy equivalence relation* (in short, *IFE*) on L if it satisfies the following conditions hold:

(i) R is reflexive, i.e.,

$$R(x,x)=(\bigvee_{y,z\in L}\mu_R(y,z),\ \bigwedge_{y,z\in L}\nu_{R(y,z)}),$$

for each $x\in L$.

(ii) R is symmetric, i.e., $R(x,y)=R(y,x)$, for any $x,y\in L$.

(iii) R is transitive, i.e., $R\circ R\subset R$.

Definition 3.2. Let R be an *IFE* on L. Then R is called an *intuitionistic fuzzy congruence* (in short, *IFC*) on L if it satisfies the following conditions hold : for any $x_1,x_2,y_1,y_2\in L$,

(i) $\mu_R(x_1+x_2,y_1+y_2)\ge\mu_R(x_1,y_1)\wedge\mu_R(x_2,y_2)$ and $\nu_R(x_1+x_2,y_1+y_2)\le\nu_R(x_1,y_1)\vee\nu_R(x_2,y_2)$.

(ii) $\mu_R(x_1x_2,y_1y_2)\ge\mu_R(x_1,y_1)\wedge\mu_R(x_2,y_2)$ and $\nu_R(x_1x_2,y_1y_2)\le\nu_R(x_1,y_1)\vee\nu_R(x_2,y_2)$.

We will denote the set of all IFCs on L as IFC(L).

Definition 3.3. Let $A\in IFI(L)$. We define a complex mapping $R_A=(\mu_{R_A},\nu_{R_A})$: $L\times L\to I\times I$ as follows : for any $x,y\in L$,

$$\mu_{R_A}(x,y)=\bigvee_{a+x=a+y}\mu_A(a)$$

and

$$\nu_{R_A}(x,y)=\bigwedge_{a+x=a+y}\nu_A(a).$$

Then R_A is called the *intuitionistic fuzzy relation induced by* A.

Lemma 3.4. Let L be a distributive lattice. If $A\in IFI(L)$, then $R_A\in IFC(L)$.

Definition 3.5. Let $R\in IFE(L)$. We define a complex mapping $A_R=(\mu_{A_R},\nu_{A_R})$: $L\to I\times I$ as follows ; for each $x\in L$,

$$\mu_{A_R}(x)=\bigwedge_{y\in L}\mu_R(xy,x)$$

and

$$\nu_{A_R}(x)=\bigvee_{y\in L}\nu_R(xy,x).$$

Then A_R is called the *intuitionistic fuzzy set in L induced by R.*

Lemma 3.6. Let L be a distributive lattice. If $R\in IFC(L)$, then $A_R\in IFI(L)$

Theorem 3.7. Let L be a distributive lattice with 0. If $A\in IFI(L)$, then $A=A_{R_A}$.

Lemma 3.8. Let L be a generalized Boolean algebra and let $A\in IFI(L)$. Then $A(x\oplus y)=R_A(x,y)$ for any $x,y\in L$.

Lemma 3.9. Let L be a lattice with 0. If $R\in IFC(L)$, then $A_{R(x)}=R(x,0)$ for each $x\in L$.

Lemma 3.10. Let $A_1,A_2\in IFI(L)$ and let $R_1,R_2\in IFC(L)$,

(1) If $A_2\subset A_1$, then $R_{A_2}\subset R_{A_1}$.

(2) If $R_2\subset R_1$, then $A_{R_2}\subset A_{R_1}$.

Theorem 3.11. Let L be a generalized Boolean algebra. If $R\in IFC(L)$, then $R_{A_R}=R$.

Theorem 3.12. Let L be a generalized Boolean algebra. Then

$$(IFI(L),\cap,+)\cong(IFC(L),\cap,+).$$

4. Products of intuitionistic fuzzy ideals

Definition 4.1. Let $A\in IFS(L_1)$ and $B\in IFS(L_2)$. We define a complex mapping $A\times B=(\mu_{A\times B},\nu_{A\times B}):L_1\times L_2\to I\times I$ as follows : for each $(x,y)\in L_1\times L_2$,

$$\mu_{A\times B}=\mu_A(x)\wedge\mu_B(y)$$

and

$$\nu_{A\times B}(x,y)=\nu_A(x)\vee\nu_B(y).$$

Then $A\times B$ is called the *product* of A and B. It is clear that $A\times B\in IFS(L_1\times L_2)$ from the above definition.

Definition 4.2. Let $A\in IFS(L_1\times L_2)$. We define two complex mappings

$$\pi_{1(A)}=(\mu_{\pi_1(A)},\nu_{\pi_1(A)}):L_1\to I\times I$$

and

$$\pi_{2(A)}=(\mu_{\pi_2(A)},\nu_{\pi_2(A)}):L_2\to I\times I.$$

as follows, respectively :

$$\pi_1(A)(x)=(\bigvee_{y\in L_2}\mu_A(x,y),\ \bigwedge_{y\in L_2}\nu_A(x,y))$$

for each $x\in L_1$ and

$$\pi_2(A)(y)=(\bigvee_{x\in L_1}\mu_A(x,y),\ \bigwedge_{x\in L_1}\nu_A(x,y))$$

for each $y\in L_2$.

Then $\pi_1(A)(x)$ and $\pi_1(A)(x)$ are called the *projections* of A on L_1 and L_2, respectively. It is clear that $\pi_1(A)\in IFS(L_1)$ and $\pi_2(A)\in IFS(L_2)$ from the above definition.

Proposition 4.3. (1) If $A_i\in IFL(L_i)$ [resp. $IFI(L_i)$] $(i=1,2)$, then $A_1\times A_2\in IFL(L_1\times L_2)$ [resp. $IFI(L_1\times L_2)$].

(2) If $A\in IFL(L_1\times L_2)$ [resp. $IFI(L_1\times L_2)$ and $IFF(L_1\times L_2)$], then $\pi_i(A)\in IFL(L_i)$ [resp. $IFI(L_i)$ and $IFF(L_i)$]. $(i=1,2)$

Definition 4.4. Let $A\in IFS(L_1\times L_2)$ and let $a\in L_2, b\in L_1$. We define tow complex mappings

$$A_1^{(a)} = (\mu_{A_1}(a), \nu_{A_2}(a)) : L_1 \to I \times I$$

and

$$A_2^{(b)} = (\mu_{A_2}(b), \nu_{A_2}(b)) : L_2 \to I \times I.$$

as follows, respectively :

$A_1^{(a)}(x) = (\mu_A(x, a), \nu_a(x, a))$ for each $x \in L_1$

and

$A_2^{(b)}(y) = (\mu_A(b, y), \nu_a(b, y))$ for each $y \in L_2$.

Then $A_1^{(a)}$ and $A_2^{(b)}$ called the *marginal intuitionistic fuzzy sets* of A (with respect to a and b). It is clear that $A_1^{(a)} \in IFS(L_1)$ and $A_2^{(b)} \in IFS(L_2)$ from the above definition.

Proposition 4.5. If $A \in IFL(L_1 \times L_2)$ [resp. $IFI(L_1 \times L_2)$ and $IFF(L_1 \times L_2)$,] then $A_1^{(a)} \in IFL(L_1)$ [resp. $IFI(L_1)$ and $IFF(L_1)$] for each $a \in L_2$ and $A_2^{(a)} \in IFL(L_2)$[resp. $IFI(L_2)$ and $IFF(L_2)$] for each $b \in L_1$.

Lemma 4.6. If $A \in IFI(L_1 \times L_2)$, then for each $a \in L_2$ and each $b \in L_1$,

$$A_1^{(a)} \times A_2^{(b)} \subset A \subset \pi_1(A) \times \pi_2(A).$$

Theorem 4.7. Let L_1 and L_2 be two lattices with 0 and let $A \in IFI(L_1 \times L_2)$. Then A is the product of an *IFI* of L_1 L_1 and of an *IFI* of L_2 if and only if

$$A_1^{(0)} \times A_2^{(0)} = \pi_1(A) \times \pi_2(A).$$

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