Intuitionistic Fuzzy Ideals on A Distributive Lattice

Kul Hur, Hee-Won Kang and Hyeong-Kee Song
Division of Mathematics and Informational Statistics, and Institute of Basic
Natural Science Wonkwang University Iksan, Chonbuk, Korea 570-749
E-mail: kulhur@wonkwang.ac.kr

Abstract

We introduce the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy congruences on a lattice, and discuss the relationship between intuitionistic fuzzy ideals and intuitionistic fuzzy congruence on a distributive lattice. Also we prove that for a generalized Boolean algebra, the lattice of intuitionistic fuzzy ideals is isomorphic to the lattice of intuitionistic fuzzy congruences. Finally, we consider the products of intuitionistic fuzzy ideals and obtain a necessary and sufficient condition for an intuitionistic fuzzy ideals on the direct sum of lattices to be representable on a direct sum of intuitionistic fuzzy ideals on each lattice.

Key words and phrases: intuitionistic fuzzy filter, intuitionistic fuzzy ideal, intuitionistic fuzzy congruence.

0. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [20], several researchers [9,15,17,18] have applied the notion of fuzzy sets to group theory. In particular, Yuan and Wu [19] have applied one to lattice theory.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues [7,8,11], and Lee and Lee[16] applied the notion of intuitionistic fuzzy sets to topology. Another researchers [3,4,12,13] applied one to group theory. In particular,
Burillo and Bastince [5,6], and Deschrijver and E. E. Kerre[10] introduced the concept of intuitionistic fuzzy relations and investigated some of its properties.

In this paper, we introduce the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy congruences on a lattice, and discuss the relationship between intuitionistic fuzzy ideals and intuitionistic fuzzy congruence on a distributive lattice. Also we prove that for a generalized Boolean algebra, the lattice of intuitionistic fuzzy ideals is isomorphic to the lattice of intuitionistic fuzzy congruences. Finally, we consider the products of intuitionistic fuzzy ideals and obtain a necessary and sufficient condition for an intuitionistic fuzzy ideals on the direct sum of lattices to be representable on a direct sum of intuitionistic fuzzy ideals on each lattice.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets $X$, $Y$ and $Z$, $f=(f_1,f_2):X\rightarrow Y \times Z$ is called a complex mapping if $f_1:X\rightarrow Y$ and $f_2:X\rightarrow Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as $I$.

**Definition 1.1[2].** Let $X$ be a nonempty set.

A complex mapping $A=(\mu_A, \nu_A):X\rightarrow I \times I$ is called an intuitionistic fuzzy set (short, IFS) on $X$ if $\mu_A+\nu_A \leq 1$, where the mapping $\mu_A:X\rightarrow I$ and $\nu_A:X\rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x\in X$ to $A$, respectively.

We will denote the set of all IFSs in $X$ as IFS($X$).

**Definitions 1.2[2].** Let $X$ be a nonempty set and let $A=(\mu_A, \nu_A)$ and $B=(\mu_B, \nu_B)$ be IFSs on $X$. Then

1. $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
3. $A^c = (\nu_A, \mu_A)$.
4. $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
5. $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
6. $\bigcup A = (\mu_A, 1-\mu_A)$, $\bigcap A = (1-\nu_A, \nu_A)$.

**Definition 1.3[5].** Let $\{A_i\}_{i \in I}$ be an arbitrary family of IFSs in $X$, where $A_i=(\mu_{A_i}, \nu_{A_i})$ for each $i \in I$. Then

(a) $\bigcap A_i = (\wedge_{\mu_{A_i}, \vee_{\nu_{A_i}}})$.
(b) $\bigcup A_i = (\vee_{\mu_{A_i}, \wedge_{\nu_{A_i}}})$.

**Definition 1.4[5].** $0_+ = (0,1)$ and $1_- = (1,0)$.

**Definition 1.5[5].** Let $X$ and $Y$ be nonempty sets and let $f:X\rightarrow Y$ be a mapping. Let $A=(\mu_A, \nu_A)$ be an IFS in $X$ and $B=(\mu_B, \nu_B)$ be an IFS in $Y$. Then

(a) the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is the IFS in $X$ defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$ and $f^{-1}(\nu_B) = \nu_B \circ f$.
(b) the image of $A$ under $f$, denoted by $f(A)$, is the IFS in $Y$ defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \left\{ \begin{array}{ll}
\bigvee_{x \in f^{-1}(y)} \mu_{A(x)} & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{if } f^{-1}(y) = \emptyset
\end{array} \right.$$

and

$$f(\nu_A)(y) = \left\{ \begin{array}{ll}
\bigwedge_{x \in f^{-1}(y)} \nu_{A(x)} & \text{if } f^{-1}(y) \neq \emptyset, \\
1 & \text{if } f^{-1}(y) = \emptyset
\end{array} \right.$$

Now, we list some concepts and results in
lattice theory.
Throughout this paper \( L = (L, +, \cdot) \) denotes a lattice.

**Definition 1.6[2].** An element \( x \in L \) is said to be relatively complemented if \( x \) is complemented in every \([a, b]\) with \( a \leq x \leq b \), i.e., \( x + y = b \) for some \( y \in [a, b] \) such that \( xy = a \). The lattice \( L \) is said to be relatively complemented if each \( x \in L \) is relatively complemented.

**Definition 1.7[2].** A relatively complemented distributive lattice with 0 is a generalized Boolean algebra.

**Definition 1.8[2].** Let \( L \) be a generalized Boolean algebra and let \( x, y \in L \). We define the difference, \( x - y \) and the symmetric difference, \( x \oplus y \), of \( x \) and \( y \), respectively as follows:
- \( x - y \) is the relative complement of \( xy \) in the interval \([0, x]\)
and
- \( x \oplus y = (x - y) + (y - x) \).
It is easily seen that:
1. \( x - y \leq x \).
2. \( y + (x - y) = x + y \).
3. \( y(x - y) = 0 \).

**Result 1.A.** [19, Lemma 3.3]. Let \( L \) be a generalized Boolean algebra. Then \( x + y = x \oplus y \oplus xy \) for any \( x, y \in L \).

**Definition 1.9[14].** A ring with 1 in which every element is idempotent is called a Boolean ring.

**Result 1.B[14, Lemma 1.9].** Let \( R \) be a Boolean ring. Then
1. \( R \) is commutative.
2. \( a + a = 0 \) for each \( a \in R \).

(1) If \((L, +, \cdot, 0)\) is a generalized Boolean algebra, then \((L, \oplus, \cdot, 0)\) is a Boolean ring.
(2) If \((R, \oplus, \cdot, 0)\) is a Boolean ring, then \((R, +, \cdot, 0)\) is a generalized Boolean algebra, where \( x + y = x \oplus y \oplus xy \). Moreover, \( x \oplus y = (x - y) + (y - x) \).

The following is the immediate result of Definition 1.9, Results 1.A, 1.B and 1.C:

**Lemma 1.10.** Let \( L \) be a generalized Boolean algebra. Then
\[ x + (x \oplus y) = y + (x \oplus y) \] for any \( x, y \in L \).

**Remark 1.10.** We can see that Lemma 1.9 is proved in Lemma 3.4 in [7, 19].

2. Intuitionistic fuzzy sublattices, ideals and filters

**Definition 2.1.** Let \( A \in IFS(L) \). (1) \( A \) is called an intuitionistic fuzzy sublattice (in short, \( IFL \)) of \( L \) if it satisfies the following conditions: for any \( x, y \in L \),
\[ \mu_A(x + y) \wedge \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \]
and
\[ \nu_A(x + y) \vee \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \].
(2) \( A \) is called an intuitionistic fuzzy filter (in short, \( IFF \)) of \( L \), if
(i) \( A \) is an \( IGF \) of \( L \),
(ii) \( A \) is monotonic, i.e., \( \mu_A(x) \leq \mu_A(y) \)
and \( \nu_A(x) \geq \nu_A(y) \) whenever \( x \leq y \).
(3) \( A \) is called an intuitionistic fuzzy ideal (in short, \( IFI \)) of \( L \), if
(i) \( A \) is an \( IFL \) of \( L \),
(ii) \( A \) is antitone, i.e., \( \mu_A(x) \geq \mu_A(y) \)
and \( \nu_A(x) \leq \nu_A(y) \) whenever \( x \leq y \).

We will denote the set of all \( IFLs \), \( IFFs \) and \( IIFs \) of \( L \) as \( IFL(L) \), \( IFF(L) \) as \( IFL(L) \), respectively.

**Proposition 2.2.** Let \( A \in IFL(L) \). Then
1. \( A \in IFF(L) \) if and only if
\[ A(xy) = (\mu_A(x) \wedge \mu_A(y)) \vee (\nu_A(x) \vee \nu_A(y)) \] for any \( x, y \in L \), i.e., \( \mu_A(x, \cdot) \cdot (I, \wedge) \) and
\( \nu_A^c : (L, \cdot) \to (I, \vee) \) are homomorphisms.

(2) \( A \in IFL(L) \) if and only if
\[
A(x + y) = (\mu_A(x) \wedge \mu_A(y), \nu_A(x) \vee \nu_A(y))
\]
for any \( x, y \in L \), i.e., \( \mu_A^c(L, +) \to (I, \wedge) \) and
\[
\nu_A^c(L, +) \to (I, \vee)
\]
are homomorphisms.

**Definition 2.3.** An intuitionistic fuzzy filter [resp. ideal] \( A \) of \( L \) is said to be prime if for any \( x, y \in L \),
\[
\mu_A(x + y) \leq \mu_A(x) \vee \mu_A(y)
\]
and
\[
\nu_A(x + y) \geq \nu_A(x) \wedge \nu_A(y)
\]
[resp. \( \mu_A(xy) \leq \mu_A(x) \vee \mu_A(y) \)
and \( \nu_A(xy) \geq \nu_A(x) \wedge \nu_A(y) \)].

**Proposition 2.4.** Let \( A \in IFL(L) \).

(1) \( A \) is an intuitionistic fuzzy prime filter of \( L \) if and only if
\[
A(xy) = (\mu_A(x) \wedge \mu_A(y), \nu_A(x) \vee \nu_A(y))
\]
and
\[
A(x + y) = (\mu_A(x) \wedge \mu_A(y), \nu_A(x) \vee \nu_A(y))
\]
for any \( x, y \in L \), i.e.,
\[
\mu_A^c(L, +, \cdot) \to (I, \vee, \wedge)
\]
and
\[
\nu_A^c(L, +, \cdot) \to (I, \wedge, \vee)
\]
are homomorphisms.

(2) \( A \) is an intuitionistic fuzzy prime filter of \( L \) if and only if
\[
A(xy) = \mu_A(x) \wedge \mu_A(y), \nu_A(x) \vee \nu_A(y)
\]
for any \( x, y \in L \), i.e., \( \mu_A^c(L, +, \cdot) \to (I, \vee, \wedge) \)
and
\[
\nu_A^c(L, +, \cdot) \to (I, \wedge, \vee)
\]
are homomorphisms.

The following is the immediate result of Proposition 2.4:

**Corollary 2.4.** Let \( A \in IFL(L) \). Then \( A \) is an intuitionistic fuzzy prime filter (resp. ideal) of \( L \) if and only if \( A^c \) is an intuitionistic fuzzy prime ideal (resp. filter) of \( L \).

**Proposition 2.5.** Let \( f : L \to L' \) be a lattice homomorphism.

(1) If \( f \) is surjective and \( A \in IFL(L) \) [resp. \( IFL(L) \) and \( IFF(L) \)], then \( f(A) \in IFL(L') \) [resp. \( IFL(L') \) and \( IFF(L') \)].

(2) If \( B \) is an intuitionistic fuzzy sublattice [resp. ideal, prime ideal, filter, and prime filter] of \( L' \), then \( f^{-1}(B) \) is an intuitionistic fuzzy sublattice [resp. ideal, prime ideal, filter, and prime filter] of \( L \).

3. Intuitionistic fuzzy congruences

**Definition 3.1.** Let \( R \in IFR(L) \). Then \( R \) is called an intuitionistic fuzzy equivalence relation (in short, IFE) on \( L \) if it satisfies the following conditions hold:

(i) \( R \) is reflexive, i.e.,
\[
R(x, x) = (\bigvee_{y \in L} \mu_R(x, y), \bigwedge_{y \in L} \nu_R(x, y)),
\]
for each \( x \in L \).

(ii) \( R \) is symmetric, i.e., \( R(x, y) = R(y, x) \), for any \( x, y \in L \).

(iii) \( R \) is transitive, i.e., \( R \circ R \subseteq R \).

**Definition 3.2.** Let \( R \) be an IFE on \( L \). Then \( R \) is called an intuitionistic fuzzy congruence (in short, IFC) on \( L \) if it satisfies the following conditions hold: for any \( x_1, x_2, y_1, y_2 \in L \),

(i) \[
\mu_R(x_1 + x_2, y_1 + y_2) \geq \mu_R(x_1, y_1) \wedge \mu_R(x_2, y_2)
\]
and
\[
\nu_R(x_1 + x_2, y_1 + y_2) \leq \nu_R(x_1, y_1) \vee \nu_R(x_2, y_2),
\]

(ii) \[
\mu_R(x_1 x_2, y_1 y_2) \geq \mu_R(x_1, y_1) \wedge \mu_R(x_2, y_2)
\]
and
\[
\nu_R(x_1 x_2, y_1 y_2) \leq \nu_R(x_1, y_1) \vee \nu_R(x_2, y_2).
\]

We will denote the set of all IFCs on \( L \) as IFC(\( L \)).

**Definition 3.3.** Let \( A \in IFL(L) \). We define a complex mapping \( R_A = (\mu_R, \nu_R) : L \times L \to I \) as follows: for any \( x, y \in L \),
\[
\mu_R(x, y) = \bigvee_{a \in A, \mu_A(a) \neq 0} \mu_A(a)
\]
and
\[ \nu_{R_A}(x, y) = \bigwedge_{a + x = a + y} \nu_A(a). \]

Then \( R_A \) is called the intuitionistic fuzzy relation induced by \( A \).

**Lemma 3.4.** Let \( L \) be a distributive lattice. If \( A \in IFL(L) \), then \( R_A \in IFC(L) \).

**Definition 3.5.** Let \( R \in IFE(L) \). We define a complex mapping \( A_R = (\mu_{A_R}, \nu_{A_R}) : L \rightarrow I \times I \) as follows: for each \( x \in L \),

\[ \mu_{A_R}(x) = \bigwedge_{y \in L} \mu_R(xy, x) \]

and

\[ \nu_{A_R}(x) = \bigvee_{y \in L} \nu_R(xy, x). \]

Then \( A_R \) is called the intuitionistic fuzzy set in \( L \) induced by \( R \).

**Lemma 3.6.** Let \( L \) be a distributive lattice. If \( R \in IFC(L) \), then \( A_R \in IFL(L) \).

**Theorem 3.7.** Let \( L \) be a distributive lattice with 0. If \( A \in IFL(L) \), then \( A = A_{R_A} \).

**Lemma 3.8.** Let \( L \) be a generalized Boolean algebra and let \( A \in IFL(L) \). Then \( A(x \oplus y) = R_A(x, y) \) for any \( x, y \in L \).

**Lemma 3.9.** Let \( L \) be a lattice with 0. If \( R \in IFC(L) \), then \( A_{R(x)} = R(x, 0) \) for each \( x \in L \).

**Lemma 3.10.** Let \( A_1, A_2 \in IFL(L) \) and let \( R_1, R_2 \in IFC(L) \).

1. If \( A_2 \subseteq A_1 \), then \( R_2 \subseteq R_A \).
2. If \( R_2 \subseteq R_1 \), then \( A_{R_2} \subseteq A_{R_1} \).

**Theorem 3.11.** Let \( L \) be a generalized Boolean algebra. If \( R \in IFC(L) \), then \( R_{A_x} = R \).

**Theorem 3.12.** Let \( L \) be a generalized Boolean algebra. Then \( (IFL(L), \cap, +) \equiv (IFC(L), \cap, +) \).

4. Products of intuitionistic fuzzy ideals

**Definition 4.1.** Let \( A \in IFS(L_1) \) and \( B \in IFS(L_2) \). We define a complex mapping \( A \times B = (\mu_{A \times B}, \nu_{A \times B}) : L_1 \times L_2 \rightarrow I \times I \) as follows: for each \( (x, y) \in L_1 \times L_2 \),

\[ \mu_{A \times B}(x, y) = \mu_A(x) \land \mu_B(y) \]

and

\[ \nu_{A \times B}(x, y) = \nu_A(x) \lor \nu_B(y). \]

Then \( A \times B \) is called the product of \( A \) and \( B \). It is clear that \( A \times B \in IFS(L_1 \times L_2) \) from the above definition.

**Definition 4.2.** Let \( A \in IFS(L_1 \times L_2) \). We define two complex mappings

\[ \pi_1(A) = (\mu_{\pi_1(A)}, \nu_{\pi_1(A)}) : L_1 \rightarrow I \times I \]

and

\[ \pi_2(A) = (\mu_{\pi_2(A)}, \nu_{\pi_2(A)}) : L_2 \rightarrow I \times I. \]

as follows, respectively:

\[ \pi_1(A)(x) = (\bigvee_{y \in L_2} \mu_A(x, y), \bigwedge_{y \in L_2} \nu_A(x, y)) \]

for each \( x \in L_1 \)

and

\[ \pi_2(A)(y) = (\bigvee_{x \in L_1} \mu_A(x, y), \bigwedge_{x \in L_1} \nu_A(x, y)) \]

for each \( y \in L_2 \).

Then \( \pi_1(A)(x) \) and \( \pi_1(A)(x) \) are called the projections of \( A \) on \( L_1 \) and \( L_2 \), respectively.

It is clear that \( \pi_1(A) \in IFS(L_1) \) and \( \pi_2(A) \in IFS(L_2) \) from the above definition.

**Proposition 4.3.** (1) If \( A_i \in IFL(L_i) \) [resp. \( IFF(L_i) \)] \((i = 1, 2)\), then \( A_1 \times A_2 \in IFL(L_1 \times L_2) \) [resp. \( IFF(L_1 \times L_2) \)].

(2) If \( A \in IFL(L_1 \times L_2) \) [resp. \( IFF(L_1 \times L_2) \)] and \( IFF(L_1 \times L_2) \), then \( \pi_i(A) \in IFL(L_i) \) [resp. \( IFL(L_i) \) and \( IFF(L_i) \)] \((i = 1, 2)\).

**Definition 4.4.** Let \( A \in IFS(L_1 \times L_2) \) and let \( a \in L_2, b \in L_1 \). We define two complex
mappings

\[ A_1^{(a)} = (\mu_{A_1}(a), \nu_{A_1}(a)) : L_1 \rightarrow I \times I \]

and

\[ A_2^{(b)} = (\mu_{A_2}(b), \nu_{A_2}(b)) : L_2 \rightarrow I \times I. \]

as follows, respectively:

\[ A_1^{(a)}(x) = (\mu_{A}(x,a), \nu_{A}(x,a)) \quad \text{for each} \quad x \in L_1 \]

and

\[ A_2^{(b)}(y) = (\mu_{A}(b,y), \nu_{A}(b,y)) \quad \text{for each} \quad y \in L_2. \]

Then \( A_1^{(a)} \) and \( A_2^{(b)} \) called the marginal intuitionistic fuzzy sets of \( A \) (with respect to \( a \) and \( b \)). It is clear that \( A_1^{(a)} \in IFS(L_1) \) and \( A_2^{(b)} \in IFS(L_2) \) from the above definition.

**Proposition 4.5.** If \( A \in IFL(L_1 \times L_2) \) 
[res. \( IFL(L_1 \times L_2) \) and \( IFF(L_1 \times L_2) \),] then 
\( A_1^{(a)} \in IFL(L_1) \) [resp. \( IFL(L_1) \) and \( IFF(L_1) \)]
for each \( a \in L_2 \) and \( A_2^{(b)} \in IFL(L_2) \) [resp. \( IFF(L_2) \)] for each \( b \in L_1. \)

**Lemma 4.6.** If \( A \in IFL(L_1 \times L_2) \), then for each 
\( a \in L_2 \) and each \( b \in L_1 \),

\[ A_1^{(a)} \times A_2^{(b)} \subset C \in \pi_1(A) \times \pi_2(A). \]

**Theorem 4.7.** Let \( L_1 \) and \( L_2 \) be two lattices with 0 and let \( A \in IFL(L_1 \times L_2) \). Then \( A \) is the product of an \( IFL \) of \( \mathcal{S}_L \cdot 1 \mathcal{S} \) \( L_1 \) and of an \( IFL \) of \( L_2 \) if and only if

\[ A_1^{(a)} \times A_2^{(b)} = \pi_1(A) \times \pi_2(A). \]

References