퍼지 랜덤 집합에 대한 중심극한정리

Central limit theorems for fuzzy random sets

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ABSTRACT

The present paper establish the improved version of central limit theorem for sums of level-continuous fuzzy random variables as a generalization of central limit theorem for sums of independent and identically distributed random sets.

Keywords: level-continuous fuzzy sets, random sets, fuzzy random variables, central limit theorems.

1. Introduction

The concept of fuzzy random variable as a natural generalization of random set was introduced by Puri and Ralescu (1986). Statistical analysis for fuzzy probability models led to the need for central limit theorems for fuzzy random variables. Klement et al. (1986) provided a good insight about the central limit theorem for fuzzy random variables assuming some Lipschitz-condition. Wu (1999) studied the completely different point of view from that by introducing the concept of weak and strong convergence in fuzzy distribution for fuzzy random variables. Kratschmer (2002) established an analogous formulation to the Lindeberg-Levy version of the central limit theorem for fuzzy random variables.

In this paper, we formulate the improved version of the above works for fuzzy random variables. It is expected that the results have considerable potential usefulness to statistical analysis for imprecise data.

2. Preliminaries

Let $K(R^p)$ be the family of all non-empty compact subsets of R^p. Then $K(R^p)$ is metrizable by the Hausdorff metric h defined by

$$h(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\},$$

A norm of $A \in K(R^p)$ is defined by

$$\| A \| = h(A, \{0\}) = \sup_{a \in A} | a | .$$

It is well known that the metric space $(K(R^p), h)$ is complete and separable (See Debreu [3]). The addition and scalar multiplication in $K(R^p)$ are defined as usual;

$$A \oplus B = \{a + b \mid a \in A, b \in B\}$$

$$\lambda A = \{\lambda a \mid a \in A\}.$$

We denote by $K_C(R^p)$ the family of convex $A \in K(R^p)$. A support function $A \in K_C(R^p)$ is defined by

$s_A : S^{p-1} \to R, \ s_A(x) = \sup_{y \in A} \langle x, y \rangle$,

where S^{p-1} denotes the unit sphere and $\langle , \rangle$ is the scalar product. It is well-known that

$s_A \in C(S^{p-1})$ and

$$s_{A \oplus B} = s_A + s_B, \quad s_{\lambda A} = \lambda s_A \text{ for } \lambda \geq 0.$$

Let $F(R^p)$ denote the space the family of all normal and upper- semicontinuous fuzzy sets u in R^p such that

$$supp\ u = cl\ \{x \in R^p : u(x) > 0\}$$

is compact. And we denote by $F_C(R^p)$ the family of fuzzy convex $u \in F(R^p)$, i.e.,

$$u(\lambda x + (1 - \lambda) y) \geq \min(u(x), u(y))$$

for $x, y \in R^p$ and $\lambda \in [0, 1]$.

For a fuzzy set u in R^p, we define the α-level set of u by

$$L_\alpha u = \begin{cases} \{x : u(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ supp\ u, & \alpha = 0. \end{cases}$$

Then it follows that $u \in F(R^p)$ (resp. $u \in F_C(R^p)$) if and only if $L_\alpha u \in K(R^p)$ (resp. $L_\alpha u \in F_C(R^p)$) for each $\alpha \in [0, 1]$.

We denote

$$L_{\alpha^+} u = cl\{x \in R^k : u(x) > \alpha\}.$$

and by $CF(R^p)$ (resp. $CF_C(R^p)$) the family of $u \in F(R^p)$ (resp. $F_C(R^p)$) such that

$$L_\alpha u = L_{\alpha^-} u \text{ for all } \alpha \in [0, 1].$$

A fuzzy sets u is called level-continuos fuzzy set if $u \in CF(R^p)$.

The addition and scalar multiplication in $F(R^p)$ are defined as usual;

$$(u \oplus v)(x) = \sup_{y+z=x} \min(u(y), v(z)),$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda), & \text{if } \lambda \neq 0 \\ I_{\{0\}}(x), & \text{if } \lambda = 0 \end{cases}$$

where $I_{\{0\}}$ is the indicator function of $\{0\}$. Then it is well-known that for each $\alpha \in [0, 1]$,

$$L_\alpha(u \oplus v) = L_\alpha u \oplus L_\alpha v$$

and

$$L_\alpha(\lambda u) = \lambda L_\alpha u.$$

A support function $u \in F_C(R^p)$ is defined by $s_u : [0, 1] \times S^{p-1} \to R, \ s_u(\alpha, x) = s_{L_\alpha u}(x)$,

Then it is well-known that

$s_{u \oplus v} = s_u + s_v, \quad s_{\lambda u} = \lambda s_u$ for $\lambda \geq 0$,

and $s_u \in C([0, 1] \times S^{p-1})$ if and only if $u \in CF_C(R^p)$.

Now, we define the metric d_∞ on $F(R^p)$ by

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha u, L_\alpha v).$$

Also, the norm of $u \in F(R^p)$ is defined as

$$\| u \| = d_\infty(u, I_{\{0\}}) = \sup_{x \in L_0 u} | x | .$$

Then it is well-known that $(F(R^p), d_\infty)$ is complete, but is not separable. (See Diamond and Kloeden [4], Klement et al. [11]. However, $(CF(R^p), d_\infty)$ is complete and separable.

3. Main Results

Let (Ω, Σ, P) be a probability space. A set-valued function $X : \Omega \to (K(R^p), h)$ is called a random set if it is measurable.

A random set X is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set X is defined by

$$E(X) = \{E(\xi) \mid \xi \in L(\Omega, R^p) \text{ and } \xi(\omega) \in X(\omega) \text{ a.s.}\},$$

where $L(\Omega, R^p)$ denotes the class of all R^p-valued random variables ξ such that $E|\xi| < \infty$.

The central limit theorem for random sets was first given, in a particular case, by Cressie [2]. The general CLT appeared in Weil [13], and independently in Gine et al. [5].

Theorem 3.1. Let $\{X_n\}$ be independent and identically distributed random sets. If $E\|X_1\|^2 < \infty$, then

$$\sqrt{n}\, h\left(\frac{X_1 \oplus \ldots \oplus X_n}{n}, E(coX_1)\right) \Rightarrow \|Z\|,$$

where Z is a centered Gaussian random element in $C(S^{p-1})$ and $\Rightarrow$ denotes the convergence in distribution.

A fuzzy valued function $X : \Omega \to F(R^p)$ is called a fuzzy random variable if for each $\alpha \in [0,1]$, $L_\alpha X$ is a random set.

A fuzzy random variable X is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded fuzzy random variable X is a fuzzy set defined by

$$E(X)(x) = \sup\{\alpha \in [0,1] \mid x \in E(L_\alpha X)\}.$$

It is well-known that if X, Y are integrably bounded, then

(1) $L_\alpha E(X) = E(L_\alpha X)$ for all $\alpha \in [0,1]$.

(2) $E(X \oplus Y) = E(X) \oplus E(Y)$.

(3) $E(\lambda X) = \lambda E(X)$.

Now we want to generalize Theorem 3.1 to the case of fuzzy random variables. Klement et al. [11] provided a generalization of the central limit theorem for fuzzy random variables taking values in the space $F_L(R^p)$ assuming of fuzzy sets $u \in F(R^p)$ such that $\alpha \mapsto L_\alpha u$ is Lipschitz, i.e., there exists a constant $M > 0$ such that

$$h(L_\alpha u, L_\beta u) \le M|\alpha - \beta| \text{ for all } \alpha, \beta \in [0,1].$$

The purpose of this paper is to generalize the result obtained by Klement et al. [11] to the case of fuzzy random variables taking values in the space $CF(R^p)$.

First we note that if a fuzzy valued function $X : \Omega \to (F(R^p), d_\infty)$ is measurable, then it is fuzzy random variable. But the converse is not true.If it is measurable (For details, see Kim [9]). Nevertheless, if we restrict our concerns to $CF(R^p)$-valued function, then they are equivalent.

Theorem 3.2. Let $\{X_n\}$ be independent and identically distributed $CF(R^p)$-valued fuzzy random variables. If $E\|X_1\|^2 < \infty$, then

(1) $$\frac{s_{X_1} + \ldots + s_{X_n} - n\, s_{E(coX_1)}}{\sqrt{n}} \Rightarrow Z,$$

(2) $$\sqrt{n}\, d_\infty\left(\frac{X_1 \oplus \ldots \oplus X_n}{n}, E(coX_1)\right) \Rightarrow \|Z\|,$$

where Z is a centered Gaussian random element in $C([0,1] \times S^{p-1})$.

The following theorem is an improvement of the above theorem.

Theorem 3.3. Let $\{X_n\}$ be independent and identically distributed $CF(R^p)$-valued fuzzy random variables. Suppose that there is a non-negative and non-decreasing function g on $[0,1]$ such that

$$Eh^2(L_\alpha X_1, L_\beta X_1) \le g(|\alpha-\beta|),$$

and

$$\int_0^1 \alpha^{-1/2} g(\alpha)^{1/2}\, d\alpha < \infty.$$

then

$$(1)\quad \frac{s_{X_1}+\ldots+s_{X_n}-n\, s_{E(coX_1)}}{\sqrt{n}} \Rightarrow Z,$$

$$(2)\quad \sqrt{n}\, d_\infty\!\left(\frac{X_1\oplus\ldots\oplus X_n}{n}, E(coX_1)\right) \Rightarrow \|Z\|.$$

Remark. For each $r\in(0,1)$, there is a sequence $\{X_n\}$ of independent and identically distributed $CF(R^p)$-valued fuzzy random variables such that

$$Eh^2(L_\alpha X_1, L_\beta X_1) \le |\alpha-\beta|^r,$$

but

$$\frac{s_{X_1}+\ldots+s_{X_n}-n\, s_{E(coX_1)}}{\sqrt{n}} \not\Rightarrow Z,$$

and

$$d_\infty\!\left(\frac{X_1\oplus\ldots\oplus X_n}{n}, E(coX_1)\right) \not\Rightarrow \|Z\|.$$

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