Proceedings of KFIS Spring Conference 2004 Volume 14, Number 1

퍼지 Beppo Levi의 정리

Fuzzy Beppo Levi's Theorem

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요 약

In this paper, we introduce Fuzzy Beppo Levi's Theorem in which we use the supremum instead of addition in the expression of Beppo Levi's Theorem. That holds under the conditions which are continuity of t-seminorm ⊤ and the fuzzy additivity of a fuzzy measure g.

1. Introduction

Sugeno [8] defined a fuzzy measure as a measure having the monotonity instead of additivity and a fuzzy integral which is an integral with respect to fuzzy measure. A generalization of the fuzzy integral was introduced by Ralescu and Adams[4]. The concept of the seminormed fuzzy integral were proposed by Suarez and Gill[6,7]. In this paper, we introduce Fuzzy Beppo Levi's Theorem in which we use the supremum instead of addition in the expression of Beppo Levi's Theorem. That holds under the conditions which are continuity of t-seminorm ⊤ and the fuzzy additivity of a fuzzy measure g.

2. PRELIMINARIES

Let X be a nonempty set, $\mathcal{A}$ be a σ-algebra of subsets of X.

A set function $g: A \to [0,1]$ is called a *fuzzy measure* if

(1) $g(\phi)=0$;

(2) $A \in \mathrm{A}$, $B \in \mathrm{A}$ and $A \subset B \Rightarrow g(A) \leq g(B)$;

(3) $A_n \subset \mathrm{A}$, $A_1 \subset A_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

$$\Rightarrow \lim_n g(A_n) = g(\bigcup_{n=1}^{\infty} A_n) ;$$

(4) $A_n \subset \mathrm{A}$, $A_1 \supset A_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$

$$\Rightarrow \lim_n g(A_n) = g(\bigcap_{n=1}^{\infty} E_n) .$$

We call $(X, \mathcal{A}, g)$ a fuzzy measure space if g is a fuzzy measure on a measurable space $(X, \mathcal{A})$

A real-valued function $h: X\rightarrow[0,1]$ is *$\mathcal{A}$-measurable* with respect to $\mathcal{A}$ and $\mathcal{B}$ (measurable, for short, if there is no confusion likely) if

$$h^{-1}(B)=\{\ x \mid h(x)\in B\ \}\in\mathcal{A} \text{ for any } B\in\mathcal{B}$$

where $\mathcal{B}$ is the σ-algebra of Borel subsets of [0, 1].

From now on, let us consider the set

$$L^0(X)=\{h: X\rightarrow[0,1] \mid h \text{ is measurable with respect to } \mathcal{A} \text{ and } \mathcal{B}\ \}$$

For any given $h\in L^0(X)$, we write $H_\alpha=\{x \mid h(x)\geq\alpha\}$, where $\alpha\in[0,1]$.

Let $A\in\mathcal{A}$, $h\in L^0(X)$. *The fuzzy integral of h on A with respect to g*, which denote by $\int_A h\, dg$, is defined by

$$\int_A h\, dg=\sup_{\alpha\in[0,1]}[\alpha\wedge g(A\cap H_\alpha)].$$

When A = X, the fuzzy integral may be denote by $\int h\, dg$.

A *t-seminorm* as a function which satisfies:

(1) For each $x\in[0,1]$, $\top(x,1)=\top(1,x)=x$

(2) For each $x_1, x_2, x_3, x_4\in[0,1]$, if $x_1\leq x_3$, $x_2\leq x_4$, then $\top(x_1,x_2)\leq\top(x_3,x_4)$.

Example 2.1 The following functions are t-seminorms

(1) $\top(x,y)=x\wedge y$

(2) $\top(x,y)=xy$

(3) $\top(x,y)=0\vee(x+y-1)$

A t-semiconorm is a function $\perp: [0,1]\times[0,1]\rightarrow[0,1]$ satisfying :

(1) For each $x\in[0,1]$, $\perp(x,0)=\perp(0,x)=x$

(2) For each $x_1, x_2, x_3, x_4\in[0,1]$, if $x_1\leq x_3$, $x_2\leq x_4$, then $\perp(x_1,x_2)\leq\perp(x_3,x_4)$.

It is easy to see that $\perp$ is a t-semiconorm if and only if there exist a t-seminorm $\top$ such that

$$\perp(x,y)=1-\top(1-x,1-y).$$

Example 2.2 The following functions are t-semiconorms

(1) $\perp(x,y)=x\vee y$

(2) $\perp(x,y)=(x+y)-xy$

(3) $\perp(x,y)=(x+y)\wedge 1$

Let $\top$ be a t-seminorm. For all $h\in L^0(X)$, the *seminormed fuzzy integral* of h over $A\in\mathcal{A}$ to fuzzy measure g is defined as :

$$\int_A h\top g=\sup_{\alpha\in[0,1]}\top(\ \alpha,\ g(A\cap H_\alpha)) \quad (2.1)$$

and the *semiconormed fuzzy integral* of h over $A\in$A to fuzzy measure g is defined as:

$$\int_A h\perp g=\inf_{\alpha\in[0,1]}\perp(\ \alpha,\ g(A\cap H_\alpha)) \quad (2.2)$$

In what follows, $\int_X h\top g$ will be denoted by $\int h\top g$ for short.

The seminormed fuzzy integral contains as a particular case the Sugeno's fuzzy integral with $\top(x,y)=x\wedge y$.

Let $\mathcal{A}$ be a collection of subsets of X. A set function g is called *fuzzy additive* on A if

$$g(\bigcup_{i\in I}A_i)=\sup_{i\in I}g(A_i)$$

for any subclass $\{A_i \mid i\in I\}$ of $\mathcal{A}$ whose union is in $\mathcal{A}$, where I is an arbitrary countable index set.

If $A_1\in\mathcal{A}$, $A_2\in\mathcal{A}$ then the fuzzy additivity of g is equivalent to the simpler requirement that

$$g(A_1\cup A_2)=g(A_1)\vee g(A_2).$$

The fuzzy additive g is called a possibility measure iff it is defined on the power set

P(X) and $g(X)=1$.

A basic probability assignment is called *consonant* iff it is defined on a nest (that is, a class fully ordered by the inclusion relation of sets).

If g is a possibility measure on the power set P(X), then its dual set function ν, which is defined by

$\nu(E)=1-g(\overline{E})$ for any $E\in$ P(X)

is called a *consonant belief* function (or *necessity measure*) on P(X), where $\overline{E}=X-E$

We recall some results given by Suarez and Alvarez [6]

Proposition 2.1 [6] If g be a possibility measure, then

(1) $\int_A (h_1 \vee h_2) \top g = \int_A h_1 \top g \vee \int_A h_2 \top g$

(2) $\int_A (h_1 \vee h_2) \perp g = \int_A h_1 \perp g \vee \int_A h_2 \perp g$

Proposition 2.2 [6] If g be a necessity measure, then

(1) $\int_A (h_1 \wedge h_2) \top g = \int_A h_1 \top g \wedge \int_A h_2 \top g$

(2) $\int_A (h_1 \wedge h_2) \perp g = \int_A h_1 \perp g \wedge \int_A h_2 \perp g$

If $\top$ and $\perp$ are continuous, we have some interesting results, like the following. Now we will introduce Fuzzy Beppo Levi's Theorem in which we use the supremum instead of addition in the expression of Beppo Levi's Theorem.

Theorem 2.3(Fuzzy Beppo Levi's Theorem)

Let $(X, \mathcal{A}, g)$ be a fuzzy measure space.

(1) If g be fuzzy additive, and $\top$ be a continuous t-seminorm, then

$$\int_A \Big(\bigvee_{n=1}^{\infty} h_n\Big) \top g = \bigvee_{n=1}^{\infty}\Big(\int_A h_n \top g\Big)$$

for $h_n \in L^0(X)$, $n=1,2,3,\cdots$.

(2) If g be fuzzy (scalar) multiplicative, and $\perp$ be a continuous t-secominorm, then

$$\int_A \Big(\bigwedge_{n=1}^{\infty} h_n\Big) \top g = \bigwedge_{n=1}^{\infty}\Big(\int_A h_n \top g\Big)$$

for $h_n \in L^0(X)$, $n=1,2,3,\cdots$.

Proof. We may assume that $A=X$ without loss of generality.

(1) Let h_n be a sequence in $L^0(X)$, $h=\bigvee_{n=1}^{\infty} h_n$, and $H_\alpha=\{x\,|\,h(x)\geq\alpha\}$.

Then $H_\alpha=\bigcup_{n=1}^{\infty} H^n_\alpha$, where $H^n_\alpha=\{x\,|\,h_n(x)\geq\alpha\}$.

Since $\top$ be a continuous t-seminorm for I_1 and I_2 are subsets of $[0,1]$,

$$\bigvee_{x\in I_1,\ y\in I_2} \top(x,y) = \top\Big(\bigvee_{x\in I_1} x,\ \bigvee_{y\in I_2} y\Big),$$

and the fuzzy additivity of g, we have

$$\begin{aligned}
\int\Big(\bigvee_{n=1}^{\infty} h_n\Big)\top g &= \sup_{\alpha\in[0,1]} \top[\ \alpha,\ g(H_\alpha)\ ] \\
&= \sup_{\alpha\in[0,1]} \top[\ \alpha,\ g(\bigcup_{n=1}^{\infty} H^n_\alpha)\ ] \\
&= \sup_{\alpha\in[0,1]} \top[\ \alpha,\ \bigvee_{n=1}^{\infty} g(H^n_\alpha)\ ] \\
&= \sup_{\alpha\in[0,1]} [\ \bigvee_{n=1}^{\infty} \top[\ \alpha,\ g(H^n_\alpha)\ ] \\
&\leq \bigvee_{n=1}^{\infty} \sup_{\alpha\in[0,1]} \top[\ \alpha,\ g(H^n_\alpha)\ ] \\
&= \bigvee_{n=1}^{\infty} \int h_n \top g. \qquad (2.3)
\end{aligned}$$

The remaining inequality can be obtained directly from the monotonicity of the seminormed fuzzy integral. In fact since $\int h_1 \top g \leq \int h_2 \top g$ for $h_1 \leq h_2$,

$$\int\Big(\bigvee_{n=1}^{\infty} h_n\Big)\top g \geq \bigvee_{n=1}^{\infty} \int h_n \top g. \qquad (2.4)$$

By (2.3) and (2.4), we have

$$\int(\bigvee_{n=1}^{\infty} h_n)\top g = \bigvee_{n=1}^{\infty} \int h_n \top g.$$

(2) Let h_n be a sequence in $L^0(X)$, $h = \bigwedge_{n=1}^{\infty} h_n$, and $H_\alpha = \{x \mid h(x) \geq \alpha\}$.

Then $H_\alpha = \bigcap_{n=1}^{\infty} H_\alpha^n$, where $H_\alpha^n = \{x \mid h_n(x) \geq \alpha\}$.

Since $\perp$ be a continuous t-semiconorm for I_1 and I_2 are subsets of $[0, 1]$,

$$\bigwedge_{x\in I_1,\ y\in I_2} \perp(x, y) = \perp(\bigwedge_{x\in I_1} x,\ \bigwedge_{y\in I_2} y),$$

and since g is necessity measure, we have

$$\begin{aligned}
\int(\bigwedge_{n=1}^{\infty} h_n)\top g &= \inf_{\alpha\in[0,1]} \perp[\ \alpha,\ g(H_\alpha)\ ] \\
&= \inf_{\alpha\in[0,1]} \perp[\ \alpha,\ g(\bigcap_{n=1}^{\infty} H_\alpha^n)\ ] \\
&= \inf_{\alpha\in[0,1]} \perp[\ \alpha,\ \bigwedge_{n=1}^{\infty} g(H_\alpha^n)\ ] \\
&= \inf_{\alpha\in[0,1]} [\ \bigwedge_{n=1}^{\infty} \top\ [\ \alpha,\ g(H_\alpha^n)\ ] \\
&\geq \bigwedge_{n=1}^{\infty} \inf_{\alpha\in[0,1]} \perp[\ \alpha,\ g(H_\alpha^n)\ ] \\
&= \bigwedge_{n=1}^{\infty} \int h_n \top g. \qquad (2.5)
\end{aligned}$$

The remaining inequality can be obtained directly from the monotonicity of the semiconormed fuzzy integral. In fact since $\int h_1 \perp g \leq \int h_2 \perp g$ for $h_1 \leq h_2$,

$$\int(\bigwedge_{n=1}^{\infty} h_n)\top g \leq \bigwedge_{n=1}^{\infty} \int h_n \top g. \qquad (2.6)$$

By (2.5) and (2.6), we have

$$\int(\bigwedge_{n=1}^{\infty} h_n)\perp g = \bigwedge_{n=1}^{\infty} \int h_n \perp g. \qquad \square$$

$\top$ and $\perp$ must be continuous; otherwise the above Theorem 2.3 does not hold.

Example 2.4 Let (X=[0, 1], $\mathcal{A}$, g) be the Lebesgue measure space.

We take

$$h(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{n+1} \\ 0, & \textit{otherwise} \end{cases},$$

and

$$\top(x, y) = \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ \frac{1}{2}, & \text{if } 0 < x < \frac{1}{2} \text{ and } 0 < y < \frac{1}{2} \\ 0, & \textit{otherwise} \end{cases}$$

Then

$$\begin{aligned}
&\int h_n \top g \\
&= \sup_{\alpha\in[0,1]} \top[\alpha, g(H_\alpha^n)] \\
&= \sup_{\alpha\in[0, \frac{1}{2}-\frac{1}{n+1}]} \top\left[\alpha, \frac{1}{2} - \frac{1}{n+1}\right] \\
&= \sup_{\alpha\in[0, \frac{1}{2}-\frac{1}{n+1}]} \top\left[\frac{1}{2} - \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1}\right] \\
&= \frac{1}{2}.
\end{aligned}$$

But

$$\begin{aligned}
&\int\left(\bigvee_{n=1}^{\infty} h_n\right)\top g \\
&= \sup_{\alpha\in[0,1]} \top\left[\alpha, g\left(\bigcup_{n=1}^{\infty} H_\alpha^n\right)\right] \\
&= \sup_{\alpha\in[0,1]} \top\left[\alpha, \bigvee_{n=1}^{\infty} g(H_\alpha^n)\right] \\
&= \sup_{\alpha\in[0, \frac{1}{2}-\frac{1}{n+1}]} \top\left[\alpha, \bigvee_{n=1}^{\infty}\left(\frac{1}{2} - \frac{1}{n+1}\right)\right] \\
&= \sup_{\alpha\in[0, \frac{1}{2}-\frac{1}{n+1}]} \top\left[\alpha, \frac{1}{2}\right] \\
&= 1.
\end{aligned}$$

Hence we have

$$\bigvee_{n=1}^{\infty} \int h_n \top g = \frac{1}{2} \neq 1 = \int\left(\bigvee_{n=1}^{\infty} h_n\right)\top g.$$

This happens because, of course, $\top$ is not continuous even if measure g is fuzzy additive.

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