ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF FUZZY RANDOM SETS

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ABSTRACT. In this paper, we establish some results on almost sure convergence for sums and weighted sums of uniformly integrable fuzzy random sets taking values in the space of upper-semicontinuous fuzzy sets in R^p.

Key words and phrases : Random sets, Fuzzy random sets, Strong law of large numbers, Uniform integrability, Tightness, Weighted sums.

1. Introduction

In recent years, strong laws of large numbers for independent fuzzy random sets can be found in the literature (e.g. [2-6, 8]). These results are generalizations of SLLN for random sets established by Artstein and Hansen [1], Taylor and Inoue [13], Uemura [15], and so on.

Pruitt [10] obtained almost sure convergence for weighted sums $\sum_{k=1}^{n} a_{nk}X_k$ of independent identically distributed real-valued random variables $\{X_n\}$ with $E|X_1|^{1+\frac{1}{\gamma}} < \infty$ by assuming

$$\max_{1\leq k\leq n} |a_{nk}| = O(n^{-\gamma}) \quad \text{for some } \gamma > 0.$$

Rohatgi [12] extended Pruitt's result to independent, but not necessarily identically distributed random variables $\{X_n\}$ by requiring stochastically bounded condition, i.e., there exists a random variable X with $E|X|^{1+\frac{1}{\gamma}} < \infty$ such that for each n,

$$P(|X_n| \geq \lambda) \leq P(|X| \geq \lambda) \quad \text{for all } \lambda > 0.$$

Taylor and Inoue [14] showed that the similar results can be obtained for random sets.

The purpose of this paper is to generalize the results of Taylor and Inoue [14] to the fuzzy case. This will be proceeded by considering uniformly integrable fuzzy random variables taking values in the space of upper-semicontinuous fuzzy sets in R^p.

2. Preliminaries

Let $\mathcal{K}(R^p)$ denote the family of non-empty compact subsets of the Euclidean space R^p. Then the space $\mathcal{K}(R^p)$ is metrizable by the Hausdorff metric h defined by

$$h(A,B) \;=\; \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

A norm of $A \in \mathcal{K}(R^p)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a\in A}|a|.$$

It is well-known that $\mathcal{K}(R^p)$ is complete and separable with respect to the Hausdorff metric h.

Let $\mathcal{F}(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;
(2) $\tilde{u}$ is upper semicontinuous;
(3) $supp\,\tilde{u} = cl\{x \in R^p : \tilde{u}(x) > 0\}$ is compact, where cl denotes the closure.

For a fuzzy subset $\tilde{u}$ of R^p, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha\tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

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Then, it follows immediately that

$$\tilde{u} \in \mathcal{F}(R^p) \text{ if and only if } L_\alpha \tilde{u} \in \mathcal{K}(R^p) \text{ for each } \alpha \in [0,1].$$

Also, if we denote $cl\{x \in R^p : \tilde{u}(x) > \alpha\}$ by $L_{\alpha^+}\tilde{u}$, then

$$\lim_{\beta \downarrow \alpha} h(L_\beta \tilde{u}, L_{\alpha^+}\tilde{u}) = 0.$$

The linear structure on $\mathcal{F}(R^p)$ is also defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0, \\ \tilde{0}(z), & \lambda = 0, \end{cases}$$

for $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$.

Now we define the metric d_∞ on $\mathcal{F}(R^p)$ as usual;

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \|L_0 \tilde{u}\| = \sup_{x \in L_0 \tilde{u}} |x|.$$

3. Main Results

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space. A set-valued function $X : \Omega \to \mathcal{K}(R^p)$ is called a random set if it is measurable. A random set X is said to be integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set X is defined by

$$E(X) = \{E(\xi) : \xi \in L(\Omega, R^p) \,\text{and}\, \xi(\omega) \in X(\omega) \, a.s.\},$$

where $L(\Omega, R^p)$ denotes the class of all R^p-valued random variables ξ such that $E|\xi| < \infty$.

A fuzzy set valued function $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is called a fuzzy random set if for each $\alpha \in [0,1], L_\alpha \tilde{X}$ is a random set. This definition was introduced by Puri and Ralescu [11] as a natural generalization of a random set. There they [11] used a term "fuzzy random variable" instead of "fuzzy random set".

A fuzzy random set $\tilde{X}$ is said to be integrably bounded if $E\|\tilde{X}\| < \infty$. The expectation of integrably bounded fuzzy random set $\tilde{X}$ is a fuzzy subset $E(\tilde{X})$ of R^p defined by

$$E(\tilde{X})(x) = \sup\{\alpha \in [0,1] : x \in E(L_\alpha \tilde{X})\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrably bounded fuzzy random sets and $\{\lambda_{ni}\}$ be a Toeplitz sequence. The problem which we will consider is to find sufficient conditions for strong convergence of weighted sums of $\{\tilde{X}_n\}$ in the following;

$$d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} co(E\tilde{X}_i)) = 0 \quad a.s.,$$

where $co(E\tilde{X}_i)$ denotes the convex hull of $E(\tilde{X}_i)$.

To this end, we require the following assumption:

(A): For each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,

$$\max_{1\le k\le m} Eh(L_{\alpha_{k-1}^+}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n) < \epsilon.$$

The next theorem implies that if $\{\tilde{X}_n\}$ is identically distributed, then it satisfies the condition (A).

Theorem 3.1. *Let $E\|\tilde{X}\| < \infty$. Then for each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,*

$$\max_{1\le k\le m} Eh(L_{\alpha_{k-1}^+}\tilde{X}, L_{\alpha_k}\tilde{X}) < \epsilon.$$

Theorem 3.2. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random sets satisfying (A). Suppose that there exists a nonnegative random variable X with $EX^{1+\frac{1}{\gamma}} < \infty$ for some $\gamma > 0$ such that for each n,*

$$P(\|\tilde{X}_n\| \geq \lambda) \leq P(X \geq \lambda) \quad \text{for all } \lambda > 0. \tag{3.1}$$

If $\{\lambda_{ni}\}$ is a Toeplitz sequence satisfying $\max_{1\leq i\leq n} |\lambda_{ni}| = O(n^{-\gamma})$, then

$$d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} co(E\tilde{X}_i)) = 0 \quad a.s.$$

It can be proved from the result of Kim [7] that if $\{\tilde{X}_n\}$ is convexly tight and $\sup_n E\|\tilde{X}_n\|^r < \infty$ for some $r > 1$, then $\{\tilde{X}_n\}$ satisfy (A) and (3.1) with $\gamma = \frac{1}{r-1}$. Thus, we obtain the following;

Corollary 3.3. *Let $\{\tilde{X}_n\}$ be a sequence of independent and convexly tight fuzzy random sets and $\sup_n E\|\tilde{X}_n\|^r < \infty$ for some $r > 1$.*

If $\{\lambda_{ni}\}$ is a Toeplitz sequence satisfying $\max_{1\leq i\leq n} |\lambda_{ni}| = O(n^{-\frac{1}{r-1}})$, then

$$d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} co(E\tilde{X}_i)) = 0 \quad a.s.$$

Note that in order to provide a SLLN by choosing $\lambda_{ni} = 1/n, 1 \leq i \leq n$; $\lambda_{ni} = 0, i > n$ in Theorem 3.2 and Corollary 3.3, we need the restrictive condition $\sup_n E\|\tilde{X}_n\|^2 < \infty$. But we can obtain much better results by similar arguments in the proof of Theorem 3.2.

Theorem 3.4. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random sets satisfying (A). Suppose that the following two conditions are satisfied;*

(1) *$\{\|\tilde{X}_n\|\}$ is uniformly integrable.*

(2) *$\sum_{n=1}^\infty \frac{1}{n^r} E\|\tilde{X}_n\|^r < \infty$ for some $1 \leq r \leq 2$.*

Then

$$\frac{1}{n} d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n co(E\tilde{X}_i)) = 0 \quad a.s.$$

Corollary 3.5. *Let $\{\tilde{X}_n\}$ be a sequence of independent and convexly tight fuzzy random sets. If* $\sup_n E\|\tilde{X}_n\|^r < \infty$ *for some* $r > 1$, *then*

$$\frac{1}{n} d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n co(E\tilde{X}_i)) = 0 \quad a.s.$$

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