Optimal System Structure of Linear Consecutive-k-out-of-n:F System

Won Young Yun, Gue Rae Kim
Dept. of Industrial Engineering
Pusan National University
Busan, KOREA
wonyun@pusan.ac.kr

Hisashi Yamamoto
Dept. of Production Information System
Tokyo Metropolitan Institute of Technology
6-6 Asahigaoka, Hino, Tokyo, JAPAN
yamamoto@cc.tmit.ac.jp

Abstract

The linear consecutive-k-out-of-n:F system consists of n components ordered linearly and fails if and only if at least k consecutive components fail. We assume that the failure times of components are independent and identical exponentially distributed. This paper develops a model to calculate the expected cost per unit time of a linear consecutive-k-out-of-n:F system. The optimization problem to find the system structure parameter k to minimize the expected cost per unit time is considered.

1. Introduction

Consecutive-k-out-of-n:F system models have been proposed for the system reliability evaluation and for the design of microwave relay stations in telecommunications, oil pipeline systems, integrated circuits, vacuum systems in accelerators, computer-ring networks, and space craft relay stations. A linear consecutive-k-out-of-n:F system consists of n components ordered linearly. The system fails if and only if at least k consecutive components fail. Figure 1 illustrates a linear consecutive-3-out-of-6:F system. Whenever the number of consecutive failures is less than 3, the signal flow from the source to the sink is not interrupted and the system works.

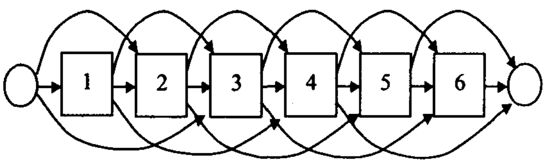

Figure 1. Linear consecutive-3-out-of-6:F system

Since the consecutive-k-out-of-n:F system was first introduced by Kontoleon [8], the system has been extensively studied by reliability engineers and researchers. Studies on this system were reviewed by Chao, Fu and Koutras [4], Chang, Cui and Hwang [1] and Kuo and Zuo [9].

One goal of reliability engineers is to increase the system reliability. If the system reliability predicted does not satisfy the reliability engineers, they have to find a reasonable method to increase it. A method has two stages:

stage 1. system design optimization
stage 2. system operation optimization

System design optimization is executed in the system design phase and include an optimal assignment problem and an economic system design problem. The optimal assignment problem includes a reliability assignment to maximize the system reliability through optimally assigning components to positions in the system. Derman, Lieberman and Ross [5], Du and Hwang [6], and Hwang and Pai [7] provided the invariant optimal design (permutation) which maximizes the system reliability. Zuo and Kuo [15] summarized the results available for the invariant optimal designs of the consecutive-k-out-of-n:F system. Another optimal design problem, economic system design, economically deals with system parameters such as p_i, k, and n. Chao and Lin [3] introduced the system structure design aspect into the problem for the consecutive-k-out-of-n:F system. Chang and Hwang [2] found the best structure, a set $\{p_1, \ldots, p_n\}$, and its assignment that maximizes the system reliability under a fixed budget. Yun and Kim [12] studied the system structure parameter k to minimize the expected cost per unit time for a consecutive-k-out-of-n:F system with $(k-1)$-step Markov dependence.

Second optimization is executed in the system operation phase in which optimal usage (optimal repair/replacement policy) is determined. The maintenance can increase system reliability. For example, we can maintain the system and the system reliability is thereby increased. Therefore, it is necessary to find an economic way to increase system reliability through maintenance. But little research on this topic exists. Recently, Yun, Kim and Yamamoto [14] and Yun, Kim and Jeong [13] introduced a maintenance design to the consecutive-k-out-of-n:F system.

We consider an economic design problem for the linear consecutive-k-out-of-n:F system with exponentially distributed components. In that case, we determine the system performance k to minimize the expected cost per unit time.

The following assumptions and notation are used in this paper.

Assumptions

1. The system consists of n identical and independent components with exponentially distributed lifetimes.
2. The failed components are replaced with new ones when the system fails.
3. Replacement cost per component $C^{(k)}$ is a increasing function for k.
4. Failures occur one at a time.
5. Replacement time is negligible.

Notation

- n number of components in the system
- k minimum number of consecutive failed components for system failure
- $T_{k|n}$ lifetime of the system
- T_i time to the ith failure $(T_0 \equiv 0)$, $i = 1, 2, \ldots, n$
- X_i time between the (i-1)th failure and ith failure; $X_i = T_i - T_{i-1}, i = 1, 2, \ldots, n$

$N_{k|n}$ number of failed components at system failure

$\binom{x}{y}$ $x!/y!(x-y)!$

$R(t), F(t)$ reliability function and failure distribution function

$S_d(\cdot)$ expected length of a cycle

$S_c(\cdot)$ expected total cost in a cycle

$CR(\cdot)$ expected cost per unit time

λ failure rate of a component

$\lfloor x \rfloor$ greatest integer lower bound of x

$A(i) = \sum_{s=0}^{\lfloor i/k \rfloor} \binom{n-i+1}{s}\binom{n-sk}{n-i}(-1)^s$

$B(i,j) = \sum_{s=0}^{j} \binom{j}{s}\binom{i+j-1-sk}{i-sk}(-1)^s$

2. Cost Model

The optimization problem we are going to tackle regards the choice of the system performance, k. The objective is to minimize the expected cost per unit time. The cost function includes the following factors:

- C_0 : cost for system breakdown
- $C^{(k)}$: replacement cost per component

If there are failed components at system failure, we must need $(C_0 + iC^{(k)})$ as the total cost, where $C^{(k)}$ is an increasing function for k.

We assume that all failed components are replaced with new ones at system failure so that we can see system failure as a renewal point. Therefore, from the well-known renewal theory, the expected cost per unit time is given by

$$CR(k) = \frac{S_c(k)}{S_d(k)} = \frac{C_0 + E(N_{k|n})C^{(k)}}{E(T_{k|n})} \quad (1)$$

Lambiris and Papastavridis [10] provided the following closed expression of the reliability of the system with i.i.d. components:

$$R(t) = \sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)\binom{i}{j} e^{-(n-i+j)\lambda t} \quad (2)$$

where

$$M = \begin{cases} n - \frac{n-1}{k} + 1 & \text{if } n+1 \text{ is a multiple of } k \\ n - \left\lfloor \frac{n+1}{k} \right\rfloor & \text{o.w.} \end{cases}$$

Using Equation 2, we have the following expression of the expected lifetime of the system of a linear consecutive-k-out-of-n:F system with i.i.d. components.:

$$E(T_{k|n}) = \int_0^\infty R(t)\,dt$$

$$= \sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)^j \binom{i}{j} \int_0^\infty e^{-(n-i+j)\lambda t} dt$$

$$= \frac{1}{\lambda}\left[\sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{1}{n-i+j}\right] \quad (3)$$

Papastavridis [11] identified the probability mass function of the number of failed components in a linear consecutive-k-out-n:F system as follows:

$$Pr(N_{k|n} = m) = \left[\sum_{x=k}^{m} \frac{2k-x}{n-m+1}\right] /$$

$$\left[\binom{n}{m-1} \cdot \sum_{i=1}^{n-x+1} \sum_{j=0}^{i^*} (B(j, i-1-j) \right.$$

$$B(m-x-j, n+j-i-m+1)) \quad (4)$$

where, $m^* = min\{m, 2k-1\}$ and $i^* = max\{0, i-2\}$.

Therefore, we can express the expected number of failed components at system failure as follows:

$$E(N_{k|n}) = \sum_{m=k}^{n} mPr(N_{k|n} = m)$$

$$= \left[\sum_{m=k}^{n} \sum_{x=k}^{m^*} \sum_{i=1}^{n-x+1} \sum_{j=0}^{i^*} 2k - x\right] \Big/$$

$$\left[\binom{n}{m} \cdot B(j, i-1-j)\right.$$

$$\left. \cdot B(m-x-j, n+j-i-m+1)\right]. \quad (5)$$

As a result, the expected cost per unit time can be expressed by the form

$$CR(k) = \lambda\left[C_0 + C^{(k)} \sum_{m=k}^{n} \sum_{x=k}^{m^*} \sum_{i=1}^{n-x+1} 2k - x\right/$$

$$\binom{n}{m} B(j, i-1-j) B(m-x-j, n+j-i-m$$

$$\Big/ \left[\sum_{i=0}^{M} A(i) \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{1}{n-i+j}\right]. \quad (6)$$

The objective is to minimize the expected cost per unit time. We can easily find the optimal system performance parameter k^* by full search, using a statistical package, for example MATHEMATICA.

3. Computational Experiments

For an example, we consider a linear consecutive-k-out-of-3:F system. The replacement cost per component $C^{(k)}$ is assumed to be linearly increasing for k, so that $C^{(k)} = C_1 + C_2 k$. Therefore, the expected cost per unit time when k=1,2,3 can derived as given below:

$$CR(1) = 3\lambda(C_0 + C_1 + C_2)$$

$$CR(2) = \left(C_0 + \frac{7}{3}(C_1 + 2C_2)\right) \Big/ \frac{7}{6\lambda}$$

$$CR(3) = (C_0 + 3(C_1 + 3C_2)) \Big/ \frac{11}{6\lambda}$$

Suppose that $\lambda = 0.02$, $C_0 = 5$, $C_1 = 1$ and $C_2 = 2$. The expected costs per unit time are 0.48, 0.2857, 0.2836 for k=1, 2 and 3, respectively. Hence the optimal k is 3.

Figure 2 shows how the expected cost per unit time varies with k when n=10 and the expected cost per unit time is not unimodal.

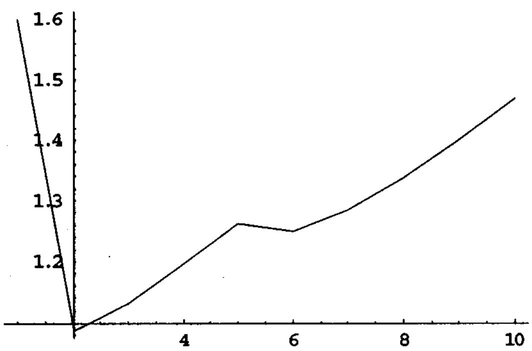

Figure 2. $CR(k)$ vs k (n=10, $\lambda = 0.02$, $C_0 = 5$, $C_1 = 1$ and $C_2 = 2$)

We studied computational experiments to evaluate the effect of various parameters on the optimal k. Let us consider the case where $\lambda = 0.02$ and

$C_1 = 1$. We solved 80 test problems with n=5,10,20,50, C_0=1,5,10,100 and C_2 =0.1,0.5,1,2,10 and the results are summarized in Table 1.

C_0/C_1	C_2/C_1	n			
		5	10	20	50
1	0.1	5	10	15	9
	0.5	4	6	2	2
	1	3	2	2	2
	2	1	1	1	1
	10	1	1	1	1
5	0.1	5	10	16	12
	0.5	5	7	4	3
	1	4	6	3	2
	2	3	2	2	2
	10	1	1	1	1

Table 1. The optimal k $(\lambda = 0.02)$

From Table 1, we found that the optimal k tends to increase with C_0/C_2 and is affected by both C_0/C_1 and C_2/C_1. Table 1 also suggests that the optimal k is n (this means the parallel system is the optimal structure) for the small system and tends to decrease with larger n.

4. Conclusion

The economic design problem was discussed for the independent model between the component in the linear consecutive-k-out-of-n:F system. For the system with independent components, we derived the expected cost per unit time and determined the optimal system performance k to minimize the expected cost per unit time. And we investigated the effect of the cost parameters on the optimal k and found that the optimal k tends to increase with C_0/C_2 as well as C_0/C_1 and C_2/C_1.

Acknowledgment

This work was supported by Grant No. R05-2002-000-00995-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

References

[1] Chang, G.J., Cui, L. and Hwang, F.K. (2000), Reliabilities of consecutive-k systems (Network Theory and Applications, Volume 4), Kluwer Academic publishers, Dordrechat, Netherlands.

[2] Chang, G.J. and Hwang, F.K.,(1988), Optimal consecutive-k-out-of-n systems under a fixed budget, Probability In Engineering and Information Sciences, 2, 63-73.

[3] Chao, M.T. and Lin, G.D.(1984), Economical design of large consecutive-k-out-of-n:F systems, IEEE Transactions on Reliability, R-33, 57-63.

[4] Chao, M.T., Fu, J.C. and Koutras, M.V.(1995), Survey of reliability studies of consecutive-k-out-of-n:F & related systems, IEEE Transactions on Reliability, 44, 120-127.

[5] Derman, C., Lieberman, G.J. and Ross, S.M.(1982), On the consecutive-k-out-of-n:F system, IEEE Transactions on Reliability, R-31, 57-63.

[6] Du, D.Z. and Hwang, F.K.(1986), Optimal consecutive-2-out-of-n Systems, Mathematics of Operations Research, 11, 187-191.

[7] Hwang, F.K. and Pai, C.K.(2000)

Sequential construction of a circular consecutive-2 System, Information Processing Letters, 75, 231-235.
[8] Kontoleon, J.M.(1980), Reliability determination of a r-successive-out-of-n: F system, IEEE Transactions on Reliability, R-29, 437.
[9] Kuo, W. and Zuo, M.J.(2002), Optimal reliability modeling: Principles and applications, John Wiley & Sons Inc., Hoboken, New Jersey, USA.
[10] Lambiris, M. and Papastavridis, S.(1985), Exact reliability formulas for linear & circular consecutive-k-out-of-n: F systems, IEEE Transactions on Reliability, R-34, 124-126.
[11] Papastavridis S.(1989), The number of failed components in a consecutive-k-out -n:F system, IEEE Transactions on Reliability, 38(3), 338-340.
[12] Yun, W.Y. and Kim, G.R.(2000), Economic design of consecutive k-out-of-n:F system, Journal of the Korean Institute of Industrial Engineers, 26, 128-135.
[13] Yun, W.Y., Kim, G.R., and Jeong, C.H.(2002), A maintenance design of connecred-(r, x)-out-of-(n, m):F system using GA, Proceedings of the 5th QMOD, 493-500.
[14] Yun, W.Y., Kim, G.R., and Yamamoto, H.(2002), Economic design of linear connected-(r,s)-out-of-(m,n):F lattice system, International Journal of Industrial Engineering, 10, 591-599.
[15] Zuo, M. and Kuo, W.(1990), Design and performance analysis of consecutive-k -out-of-n structure, Naval Research Logistics, 37, 203-230.