The Virtual Waiting Time of the $M/G/1$ Queue with Customers of n Types of Impatience*

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Abstract

We consider $M/G/1$ queue in which the customers are classified into $n+1$ classes by their impatience time. First, we analyze the model of two types of customers;one is the customer with constant impatience duration k and the other is patient customer. The expected busy period of the server and the limiting distribution of the virtual waiting time process are obtained. Then, the model is generalized to the one in which there are n classes of customers according to their impatience duration.

Keywords: $M/G/1$ queue; impatient customers; virtual waiting time; limiting distribution

1 Introduction

In this paper, the $M/G/1$ queue with customers of several types of impatience is considered. The customers who arrive according to Poisson process of rate ν are classified into $n+1$ classes by their impatience time. Their impatience times are $k_1, k_2, ..., k_n$ and ∞. The customer whose impatience time is ∞ means patient customer, that is, the one who waits until his/her service starts however long the waiting time is. Let the proportion of customers with impatience time k_i be p_i, $i = 1, 2, ..., n$, and ∞ be q. We assume that $k_1 < k_2 < \cdots < k_n$, and $0 < p_i \leq 1$ and $0 \leq q < 1$. The service times of all customers are independent and identically distributed with distribution function G and mean m. There is one server in the system and the service discipline is FIFO.

Under this model, we analyze the virtual waiting process. First, the formula for the expected busy period of the server is derived and then the limiting distribution of the process is obtained by calculating the expected number of downcrossings of the process during a busy period for a given level. This study

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is the extension of Bae et al.(2001), the model in which is the special case of our model - $n = 1$, $p_1 = 1$, and $q = 0$. In this paper, more general formulas are given and the assumption on G is weaker than in Bae et al.(2001).

Let Z_t denotes the virtual waiting time process and ρ_i denote the traffic intensity while $k_i < Z_t < k_{i+1}$, $i = 0, 1, 2, \ldots$ with $k_0 = 0$ and $k_{n+1} = \infty$. Then, $\rho_0 = \nu m$, $\rho_i = \nu m(1 - \sum_{j=1}^{i} p_j)$, and $\rho_0 > \rho_1 > \cdots > \rho_n = q\nu m \geq 0$. For the stability of the system, it is assumed that $\rho_n < 1$. For $x \in (a, b]$, we define the first exit time $T_a^b(x; \rho) = \min\{t \geq 0 | Z_t \notin (a, b]\}$ given that $Z_0 = x$ and the traffic intensity while $Z_t \in (a, b]$ is ρ. We let $G_e(x) = (1/m)\int_0^x (1 - G(s))ds$ be the equilibrium distribution function of G and denote the Stieltjes-convolution of F and G by $(F * G)(x) = \int_{0-}^{x} G(x - s)dF(s)$ and the n-fold recursive Stieltjes-convolution of F by $F^{n*}(x)$ with $F^{0*}(x) = I_{\{x \geq 0\}}$.

2 $M/G/1$ with Patient and Impatient Customers

We first consider the case of $n = 1$. The proportion of impatient customers is $p_1 = 1 - q$ and their impatience time is k. Unless confused, we use $T(x)$, $0 < x \leq k$, instead of $T_0^k(x; \rho_0)$.

2.1 Expected Busy Period

Define $P(s, x) = \Pr\{Z_{T(x)} > k + s\}$, $s \geq 0$. Then, by Bae et al. (2001),

$$P(s, x) = \frac{(H * \tilde{G}_s)(k)}{H(k)} H(k - x) - (H * \tilde{G}_s)(k - x), \tag{1}$$

where $H(x) = H_{\rho_0}(x) = \sum_{n=0}^{\infty} \rho_0^n G_e^{n*}(x)$ and $\tilde{G}_s(x) = \nu \int_0^x (1 - G(y + s))dy$. Throughout this section, $H_{\rho_0}(x)$ is often abbreviated by $H(x)$.

Let $\tau(x)$ be the busy period when $Z_0 = x$, $x \geq 0$. Then, by the Markov property of Z_t, wee see that

$$\tau(x) = T(x) + 1_{\{Z_{T(x)} > k\}} \{Y + \tau(k)\}, \quad 0 \leq x \leq k,$$

hence, for $0 \leq x \leq k$,

$$E[\tau(x)] = E[T(x)] + P(0, x) \left\{ \frac{E[Z_{T(x)} - k | Z_{T(x)} > k]}{1 - \rho_1} + E[\tau(k)] \right\}. \tag{2}$$

Lemma 1 *For* $0 < x \leq k$,

$$\begin{aligned} & E[Z_{T(x)} - k |\, Z_{T(x)} > k] \\ &= \frac{kH(k - x) - (k - x)H(k)}{H(k) - H(k - x)} + (1 - \rho_0) \frac{H(k) \int_0^{k-x} H(u)\,du - H(k - x) \int_0^k H(u)\,du}{H(k) - H(k - x)}. \end{aligned}$$

Proof

The lemma can be proved after observing

$$E[Z_{T(x)} - k |\, Z_{T(x)} > k] = \int_0^\infty \frac{\Pr\{Z_{T(x)} > k + s\}}{\Pr\{Z_{T(x)} > k\}} ds = \int_0^\infty \frac{P(s, x)}{P(0, x)} ds.$$

Lemma 2 *For* $0 < x \leq k$,

$$E[T(x)] = \frac{H(k-x)}{H(k)} \int_0^k H(u)\,du - \int_0^{k-x} H(u)\,du.$$

Proof
Using Optional Stopping Theorem of martingale, we can prove the lemma.

Putting $x = k$ in equation (2) gives

$$E[\tau(k)] = \frac{1}{1-\rho_1}\left\{(\rho_0 - \rho_1)\int_0^k H(u)du + k\right\},$$

and substituting the above equation into equation (2) yields

$$E[\tau(x)] = \frac{x + (\rho_0 - \rho_1)\int_{k-x}^k H(u)du}{1-\rho_1}, \quad 0 < x \leq k,$$

and

$$E[\tau(k+s)] = \frac{s}{1-\rho_1} + E[\tau(k)] = \frac{k + s + (\rho_0 - \rho_1)\int_0^k H(u)du}{1-\rho_1}.$$

Now, some calculations by conditioning on the starting level x yield the next theorem.

Theorem 1 *The formula for the expected busy period of the queue is given by*

$$E[\tau] = \frac{m(1-q)}{1-\rho_1}H(k) + \frac{mq}{1-\rho_1}.$$

2.2 Expected Number of Downcrossings

Let D^z be the number of downcrossings of level z during a busy period, and $D^z(x)$ be that of level z during a busy period given that $Z_0 = x, (x > 0)$. Then, for $0 < z \leq k$, by the same argument as that of Bae et al.(2001),

$$E[D^z(x)] = \begin{cases} H(z) - H(z-x) & \text{if } 0 < x \leq z, \\ H(z) & \text{if } x > z, \end{cases}$$

and hence,

$$E[D^z] = H'(z)/\nu.$$

For $z > k$, if $k < x \leq z$,

$$E[D^z(x)] = \Pr\{Z_{T_k^z(x;\rho_1)} > z\}\,(1 + E[D^z(z)]) + \Pr\{Z_{T_k^z(x;\rho_1)} = k\}E[D^z(k)].$$

That is,

$$E[D^z(x)] = \left(1 - \frac{H_{\rho_1}(z-x)}{H_{\rho_1}(z-k)}\right)(1 + E[D^z(z)]) + \frac{H_{\rho_1}(z-x)}{H_{\rho_1}(z-k)}E[D^z(k)]. \quad (3)$$

By putting $x = z$, we have

$$E[D^z(z)] = E[D^z(k)] + H_{\rho_1}(z-k) - 1$$

Substituting $E[D^z(z)]$ in the above equation into equation (3) yields

$$E[D^z(x)] = E[D^z(k)] + H_{\rho_1}(z-k) - H_{\rho_1}(z-x). \tag{4}$$

For $z > k$, if $x > z$,

$$E[D^z(x)] = 1 + E[D^z(z)] = E[D^z(k)] + H_{\rho_1}(z-k). \tag{5}$$

For $z > k$, if $0 < x \le k$, we first denote the distribution function of $T_0^k(x;\rho_0)$ by $F_0^k(y;x,\rho_0)$ or simply $F(y;x)$. Note that $T_0^k(x;\rho_0)$ has a mass of $H_{\rho_0}(k-x)/H_{\rho_0}(k)$ at 0 and continuous distribution in (k,∞). Namely,

$$F(y;x) = \begin{cases} 0 & \text{if } y < 0, \\ H_{\rho_0}(k-x)/H_{\rho_0}(k) & \text{if } 0 \le y \le k, \\ 1 - P(y-k;x) & \text{if } y \ge k, \end{cases}$$

where the formula of $P(s;x)$ is in equation (1).

Now, we can derive $E[D^z(x)]$ for $z > k, 0 < x \le k$ by conditioning on $T(x)$

$$\begin{aligned} E[D^z(x)] &= \Big(H_{\rho_1}(z-k) + E[D^z(k)]\Big)\left(1 - \frac{H_{\rho_0}(k-x)}{H_{\rho_0}(k)}\right) \\ &\quad -(F * H_{\rho_1})(z;x) + \frac{H_{\rho_1}(z)H_{\rho_0}(k-x)}{H_{\rho_0}(k)}, \end{aligned}$$

where $(F * H_\rho)(z;x) = \int_{0-}^{z} H_\rho(z-y)\, d_yF(y;x)$. Letting $x = k$ gives us

$$E[D^z(k)] = H_{\rho_1}(z-k)\Big(H_{\rho_0}(k) - 1\Big) - H_{\rho_0}(k)(F * H_{\rho_1})(z;k) + H_{\rho_1}(z),$$

and it follows that

$$E[D^z(x)] = \Big(H_{\rho_0}(k) - H_{\rho_0}(k-x)\Big)\Big(H_{\rho_1}(z-k) - (F*H_{\rho_1})(z;k)\Big) + H_{\rho_1}(z) - (F*H_{\rho_1})(z;x).$$

After substituting the above equation into equation (4) and equation (5), we get the following summary: when $z > k$, $E[D^z(x)]$ is given by

$$\Big(H_{\rho_0}(k) - H_{\rho_0}(k-x)\Big)\Big(H_{\rho_1}(z-k) - (F*H_{\rho_1})(z;k)\Big) + H_{\rho_1}(z) - (F*H_{\rho_1})(z;x), \quad \text{if } 0 < x \le k,$$

$$H_{\rho_0}(k)\Big(H_{\rho_1}(z-k) - (F * H_{\rho_1})(z;k)\Big) + H_{\rho_1}(z) - H_{\rho_1}(z-x), \quad \text{if } k < x \le z,$$

and

$$H_{\rho_0}(k)\Big(H_{\rho_1}(z-k) - (F * H_{\rho_1})(z;k)\Big) + H_{\rho_1}(z), \quad \text{if } x > z.$$

Therefore, we have that for $z > k$,

$$\begin{aligned} E[D^z] &= (H'_{\rho_0}(k)/\nu)\Big(H_{\rho_1}(z-k) - (F * H_{\rho_1})(z;k)\Big) \\ &\quad + H_{\rho_1}(z) - \int_0^k (F * H_{\rho_1})(z;x)\, dG(x) - \int_k^z H_{\rho_1}(z-x)\, dG(x) \end{aligned}$$

Now, the limiting probability density function $f(z)$ of the virtual waiting time process is calculated by Brill and Posner(1977) as followings:

$$f(z) = E[D^z]/E[\tau], \quad z > 0,$$

and the limiting distribution has a mass $1 - \int_0^\infty f(z)dz$ at 0.

3 $M/G/1$ with Patient and n types of Impatient Customers

3.1 Expected Busy Period and Number of Downcrossings

Given that $Z_0 = x$, let $E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$ be the expected busy period of the queue and $E[D^z(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$ be the expected number of downcrossings of level z during a busy period. In the previous section, we have obtained $E[\tau(x; k_1; \rho_0, \rho_1)]$ and $E[D^z(x; k_1; \rho_0, \rho_1)]$. In this section, it is provided the recursive method of deriving $E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$ and $E[D^z(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$.

Assume that $E[\tau(x; k_1, ..., k_j; \rho_0, ..., \rho_j)]$ is known for $j = n-1$. When $j = n$, for $x > k_1$ the busy period of the queue starting at x is equal in distribution to the sum of the time to reaching k_1 of Z_t and the busy period starting at k_1. Hence, $E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$, abbreviated by $E[\tau(x)]$, can be expressed as

$$E[\tau(x)] = E[\tau(x - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)] + E[\tau(k_1)]. \tag{6}$$

For $0 < x \le k_1$, by conditioning on $T_0^{k_1}(x; \rho_0)$, $E[\tau(x)]$ is represented as

$$\begin{aligned} E[\tau(x)] \;=\; & \int_{k_1}^{\infty} E[\tau(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; x, \rho_0) \\ & + \frac{H_{\rho_0}(k_1 - x)}{H_{\rho_0}(k_1)} \int_0^{k_1} H_{\rho_0}(u)du - \int_0^{k_1 - x} H_{\rho_0}(u)du + E[\tau(k_1)] \left(1 - \frac{H_{\rho_0}(k_1 - x)}{H_{\rho_0}(k_1)}\right). \end{aligned}$$

Putting $x = k_1$ gives us

$$E[\tau(k_1)] = \int_0^{k_1} H_{\rho_0}(u)du + H_{\rho_0}(k_1) \int_{k_1}^{\infty} E[\tau(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; k_1, \rho_0)$$

Using this, we get that for $0 < x \le k_1$

$$\begin{aligned} & E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)] \\ & = \int_0^{k_1} H_{\rho_0}(u)du - \int_0^{k_1 - x} H_{\rho_0}(u)du \\ & \quad + \int_{k_1}^{\infty} E[\tau(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; x, \rho_0) \\ & \quad + (H_{\rho_0}(k_1) - H_{\rho_0}(k_1 - x)) \int_{k_1}^{\infty} E[\tau(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; k_1, \rho_0), \end{aligned}$$

and from equation (6), for $x > k_1$

$$
\begin{aligned}
&E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)] \\
&= E[\tau(x - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)] + \int_0^{k_1} H_{\rho_0}(u)du \\
&\quad + H_{\rho_0}(k_1) \int_{k_1}^{\infty} E[\tau(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; k_1, \rho_0).
\end{aligned}
$$

Therefore, $E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$ can be obtained inductively.

$E[D^z(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$, simply $E[D^z(x)]$, is also derived similarly to $E[\tau(x; k_1, ..., k_n; \rho_0, ..., \rho_n)]$. First, observe that for $0 < z \leq k_n$

$$E[D^z(x; k_1, ..., k_n; \rho_0, ..., \rho_n)] = E[D^z(x; k_1, ..., k_{n-1}; \rho_0, ..., \rho_{n-1})].$$

When $z > k_n$ and $x > k_1$,

$$E[D^z(x)] = E[D^{z-k_1}(x - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)] + E[D^z(k_1)], \quad (7)$$

and when $z > k_n$ and $0 < x \leq k_1$,

$$
\begin{aligned}
E[D^z(x)] &= \int_{k_1}^{\infty} E[D^{z-k_1}(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; x, \rho_0) \\
&\quad + E[D^z(k_1)] \left(1 - \frac{H_{\rho_0}(k_1 - x)}{H_{\rho_0}(k_1)}\right).
\end{aligned}
$$

After letting $x = k_1$ in the above equation, we can get for $z > k_n$ and $0 < x \leq k_1$

$$
\begin{aligned}
&E[D^z(x)] \\
&= (H_{\rho_0}(k_1) - H_{\rho_0}(k_1 - x)) \int_{k_1}^{\infty} E[D^{z-k_1}(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; k_1, \rho_0) \\
&\quad + \int_{k_1}^{\infty} E[D^{z-k_1}(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; x, \rho_0),
\end{aligned}
$$

and from equation (7), for $z > k_n$ and $x > k_1$

$$
\begin{aligned}
E[D^z(x)] &= E[D^{z-k_1}(x - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)] \\
&\quad + H_{\rho_0}(k_1) \int_{k_1}^{\infty} E[D^{z-k_1}(y - k_1; k_2 - k_1, ..., k_n - k_1; \rho_1, ..., \rho_n)]\, d_y F_0^{k_1}(y; k_1, \rho_0).
\end{aligned}
$$

Finally, the limiting distribution of Z_t can be calculated by obtaining the unconditioned expected busy period and the number of downcrossings.

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