Confounded Row-Column Designs

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Abstract

Confounded row-column designs for factorial experiments are studied in this paper. The Designs, thus, have factorial balance with respect to estimable main effects and interactions. John and Lewis (1983) considered generalized cycle row=column designs for factorial experiments. A simple method of constructing confounded designs using the classical method of confounding for block designs is described in this paper.

1. Introduction

Block designs for factorial experiments have been extensively investigated in the literature over the last several decades. In some experiments, however, two sources of extraneous variability must be controlled where Latin square and Youden square designs are quite useful. A more flexible class of two-dimensional designs is provided by row-column designs as Latin squares and squares do not always suit the requirements of the experimenter. Yates (1937), and Rao (1946) gave some row-column designs for 2-level and 3-level factorial experiments. The algorithmic methods of Patterson (1976), Bailey et al. (1977), and Patterson and Bailey (1978) are quite general in that they are capable of producing designs with various blocking structures, including row-column designs. John and Lewis (1983) developed the generalized cyclic method of construction for obtaining row-column designs. They also derived main effect and interaction efficiency factors for generalized cyclic designs, and gave some guidelines for choosing the row and column component designs appropriately.

It is clear that the developments in the area of row-column designs for factorial designs are relatively few. The purpose of this paper, therefore, is to present a simple method of obtaining confounded row-column designs based on the classical method of confounding. Thus, the method is applicable to symmetrical factorial experiments, with number of levels a prime or a prime power. The model and some preliminaries of row-column designs are presented in Section 2. The method of construction is presented in Section 3. The method is illustrated with the help of several examples.

2. Model and Preliminaries

Consider a row-column design D having v treatments, p rows and q columns. Let r denote the constant number of treatments replications The model for a row-column design

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is given by,

$$Y_{ijk} = \mu + r_i + \rho_j + r_k + \epsilon_{ijk}$$

where Y_{ijk} is the observation from the jth row, kth column to which the i th treatments has been applied, μ is the overall mean, r_i, ρ_j, r_k are the effects of the i th treatment, jth row, and kth column respectively, and $\epsilon' s$ are uncorrelated random errors with zero and variance σ^2.

Let N_1 and N_2 denote the treatment-row and treatment-column incidence matrices respectively. The component designs corresponding to N_1 an N_2 will be denoted by D_1 and D_2 respectively. The intra-block reduced normal equations for estimating the vector of treatment parameters $r = (r_1, r_2, \cdots, r_v)', r_i$ being the i th treatment effect, are given by $Cr = Q$ with

$$C = rI - \frac{1}{q}N_1N_1' - \frac{1}{p}N_2N_2' + \frac{R^2}{pq}11',$$

where I denotes the identity matrix and 1 denotes the column vector of 1's of size v each. It is easy to verify that

$$C = C_1 + C_2 - C_b \tag{2.1}$$

where

$$C_1 = rI - \frac{1}{q}N_1N_1'$$

$$C_2 = rI - \frac{1}{p}N_2N_2'$$

$$C_b = r(I - \frac{1}{v}11')$$

are the intra-block information matrices of the component designs D_1, D_2 and D_b respectively.

Here D_b denotes a design having one block obtained by ignoring the row and column classifications, with the incidence matrix $N_b = r1$.

Lemma 2.1. Suppose $u = (u_1, u_2, \cdots, u_v)'$ with $u'1 = 0$ is an eigen vector of both C_1 and C_2.

(a) If $u'r$ is confounded in the component design $D_1(D_2)$ and it is not confounded in the component deign $D_2(D_1)$, then it is confounded in the row-column design D.

(b) If $u'r$ is unconfounded in both the component designs D_1 and D_2, then it is unconfounded in the row-column design D as well.

Proof. (a) clearly $u'r$ is unconfounded in D_b. Suppose it is confounded in the component design D_1 only. Then, using $C_1u = 0$, $C_2u = C_bu = ru$ in equation (2.1) we get $Cu = 0$. (b) Now suppose $u'r$ is unconfounded in both D_1 and D_2. Then , $C_1u = C_2u = C_bu = ru$. Thus, $Cu = ru$. Hence the lemma.

3. The Method of Construction

As mentioned in the introduction section, attention will be restricted to equireplicate s^m factorial experiments involving m factors $F_1, F_2, \cdots, F_m$, having s levels each. The treatment combinations will be denoted by n-tupkes $a_1a_2 \cdots a_n$, n. Consider the row-column design D for a s^m factorial experiment. Let $p = s^{m1}, q = s^{m2}$, with $pq = s^{m1+m2} = rs^m$. Thus, experiment. Let $p = s^{m1}, q = s^{m2}$, with $pq = s^{m1+m2} = rs^m$. Thus,

$$r = s^{m1+m2-m}.$$

Since $r \geq 1$, we have,

$$m_1 + m_2 \geq m.$$

Let $A_1, A_2, \cdots, A_{m-m_2}$ denotes the independent interactions confounded between s^{m_1} rows of D. Then, a total of $s^{m-m_2} - 1$ treatment degrees freedom are onfonded between $p = s^{m_1} = rs^{m-m_2}$ rows of D. The total number of effects confounded between rows, each effect having $s-1$ degrees of freedom, are then given by $(s^{m-m_2}-1)/(s-1)$. Thus, the number of generalized interactions confounded between rows of D is giver by

$$g_1 = \frac{s^{m-m_2}-1}{s-1} - (m - m_2).$$

Let these generalized interactions be denoted by $A_{m-m_2+1}, A_{m-m_2+2}, \cdots, A_{m-m_2+g_1}$ The factorial effects $A_1, A_2, \cdots, A_{m-m_2+g}, B_1, B_2, \cdots, B_{m-m_1+g_2}$ are to be chosen such that they are all distinct.

We first constrict the key or the principal block for rows by confounding $m - m_2$ independent interactions $A_1, A_2, \cdots, A_{m-m_2}$ between rows of D. Let this row key block be denoted by $(a_1^i a_2^i \cdots a_n^i, i = 1, 2, \cdots, s^{m_2})$ where $a_\ell^i \epsilon \{0, 1, \cdots, s-1\}$ with $a_\ell^i = 0, \ell = 1, 2, \cdots, n$. Next the column key block is similarly obtained by confounding $m - m_1$, independent interactions $B_1, B_2, \cdots, B_{m-m_1}$ between columns of D. Let the column key block key block be denoted by $b_1^j b_2^j \cdots b_n^j, j = 1, 2, \cdots, s^{m_1})$, where $b_\ell^j \epsilon (0, 1, \cdots, s-1)$ with $b_\ell^1 = 0, \ell = 1, 2, \cdots, n$. Then the treatment combination in the

ℓ_1^{th} row and ℓ_2^{th} column of D is given by $c_1, c_2, \cdots, c_n$, with $c_\ell = a_\ell^{\ell_2} + b_\ell^{\ell_1}$, $\ell = 1, 2, \cdots, n$, where the addition is done $mod(d)$.

Example 2.1 A 2^4 experiment using a row-column design with $p = q = 2^2$. Thus $s = 2, m = 4, m_1 = m_2 = 2, m - m_1 = m - m_2 = 2$. Let $A_1 = F_1F_2, A_2 = F_3F_4$, $B_1 = F_1F_2F_3$, $B_2 = F_2F_3F_4$. Then $A_3 = F_1F_2F_3F_4$, $B_3 = F_1F_4$. The row and column key blocks and the resulting row-column design are given below.

Column Key Block	Row Key Block 0000 1100 0011 1111
0000	0000 1100 0011 1111
0110	0110 1010 0101 1001
1101	1101 0001 1110 0010
1011	1011 0111 1000 0100

The above single replicate design completely confounds the interactions $F_1F_2, F_3F_4, F_1F_2F_3F_4, F_1F_2F_3, F_2F_3F_4$ and F_1F_2.. All the main other interactions are estimated with full efficiency. A design which completely confounds only the four-factor interaction $F_1F_2F_3F_4$ can be constructed by adding a second replicate, using $A_1 = F_1F_3, A_2 = F_1F_3, B_1 = F_1F_3F_4, B_2 = F_1F_2F_4$.

Example 2.2 A 2^4 experiment using a row-column design with $p = 2^2$, $q = 2^3$. Here $s = 2, m = 4$, $m_1 = 2, m_2 = 3$, $m - m_1 = 2$, $m - m_2 = 1$. Let $A_1 = F_1F_2F_3F_4$, $B_1 = F_1F_2F_3$, $B_2 = F_2F_3F_4$. Then, $B_3 = F_1F_4$. The row and column key blocks and the resulting row-column design are given below.

Column Key Block	Row Key Block							
	0000	0011	0101	0110	1100	1010	1001	1111
0000	0000	0011	0101	0110	1100	1010	1001	1111
0110	0110	0101	0011	0000	1010	1100	1111	1001
1101	1101	1110	1000	1011	0001	0111	0100	0010
1011	1011	1000	1110	1101	0111	0001	0010	0101

Note that each treatment is replicated $r = s^{m_1 + m_2 - m} = 2$ times in the above design. The interactions F_1F_4, $F_1F_2F_3$, $F_2F_3F_4$ and $F_1F_2F_3F_4$. are completely confounded, and there is no loss of information on any of the main effects and other interactions.

Example 2.3 A 3^3 experiment, with $p=3, q=3^2$. Here $s=3, m=3, m_1=1, m_2=2$, $m-m_1=2, m-m_2=1$. Let $A_1=F_1F_2F_3$, $B_1=F_1F_2F_3^2$, $B_2=F_2F_3$. Then $B_3=F_1F_2^2$, $B_4=F_1F_3$. The design is then as follows.

Column	Row Key Block								
Key Block	000	102	012	201	021	111	120	210	222
000	000	102	012	201	021	111	120	210	222
112	112	211	121	010	100	220	202	022	001
221	221	020	200	122	212	002	011	101	110

The above single replicate design completely confounds the interactions $F_1F_2F_3$, $F_1F_2F_3^2, F_2F_3, F_1F_2^2, F_1F_3$, having degrees of freedom each. All other factorial effects are estimated with full efficiency.

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